A Theory of Equivalent Expectation Measures for Contingent Claim Returns

SANJAY K. NAWALKHA (University of Massachusetts Amherst)
XIAOYANG ZHUO (Beijing Institute of Technology)

UMassAmherst | Isenberg School of Management

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The Motivation: Risk and Return Analysis of Contingent Claims

- Under incomplete markets, not all risks can be traded and hedged, and so the market portfolio may not be optimal for all investors.
- Even in complete markets, due to heterogenous beliefs and differences in preferences, different investors may hold different optimal portfolios, which are significantly different from the representative investor's market portfolio.
- Mean variance efficiency may be useful even with a subset of assets under certain arbitrage-based models, such as Ross (1976) APT or certain arbitrage-based pricing kernel models—even with arbitrary preferences and general asset return distributions.

The Motivation: Risk and Return Analysis of Contingent Claims

- Due to market frictions, regulatory constraints on the movement of international cash flows, etc., mean variance efficient portfolios may need to be computed over a subset of assets.
- Due to a need for specialization of skills, many money fund managers hold specialized portfolios, such as Treasury bond portfolios, corporate bond portfolios, equity portfolios. These managers are often evaluated based upon portfolio performance measures, such as the Sharpe ratio, which only use mean and variance.

The Motivation: Risk and Return Analysis of Contingent Claims

- **The big problem:** the investor wishes to optimize over all kinds of asset classes, such as Treasury bonds, corporate bonds, equities, commodities, etc., each of which follow different types of asset return processes with stochastic volatility, jumps, etc.
- **The smaller problem:** the investor wishes to compute a mean-variance efficient portfolio with a limited number of asset classes, with some limitations on the types of processes allowed.

The Motivation: Risk and Return Analysis of Contingent Claims

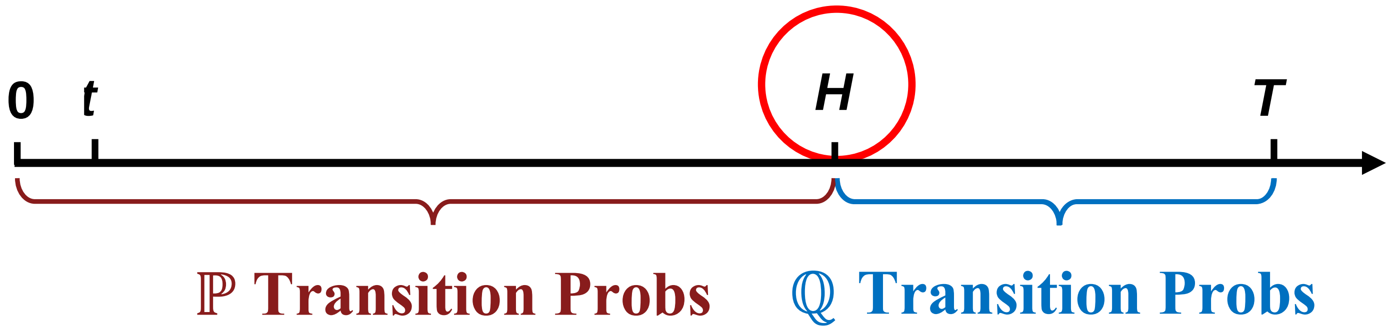
- With continuing advances in theoretical asset pricing and continuing efficiency gains in computing power, **the big problem** may be solved in the in intermediate to long-term future.
- In this paper, we address **the smaller problem**: how to compute mean-variance efficient portfolios using a given asset class or at most two or three asset classes, with some limitations on the stochastic processes for their returns.
- We take it as a given that an investor holds a subset of assets using which they wish to compute a mean-variance efficient portfolio over a finite horizon H.

The Motivation: Risk and Return Analysis of Contingent Claims

- Even after 70 years of the discovery of Modern Portfolio Theory by Markowitz (1952), even **the smaller problem** as been solved only for publicly-issued equities from large equity indices, such as the S & P 500.
- One may be surprised to know that parsimonious formulas for expected returns, variances, and covariances over a finite horizon *H*, **do not exist** for the most basic claims, such as European call options on equites, Treasury bonds, corporate bonds, using even the most basic models, such as Black-Merton-Scholes, Vasicek (1973), and Merton (1974), respectively.

How to Compute Expected Price

The $\mathbb{R}$ Measure



How to Compute Expected Return

The $\mathbb{R}$ measure allows an **analytical solution of the expected price of a contingent claim at a finite horizon H.** For example,

$\mathrm{E}_t\left[C_H\right]$ = This is what we do in the 2022 JF paper

Why is this a big deal? It applies to hundreds of models and zillions of securities. Rubinstein solution of expect future price applies only to a European call option under the Black-Scholes model.

How to Compute Expected Return

- How does the $\mathbb{R}$ measure work for the **expected price**?

$$\begin{aligned}\mathbb{E}_t\left[C_H\right] &= \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\right]\right] \\ &= \mathbb{E}_t^{\mathbb{R}}\left[\mathbb{E}_H^{\mathbb{R}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\right]\right] \\ &= \mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\right]\end{aligned}$$

- Look carefully: the **change of measure occurs at a future time *H***, and not at time zero or the current time *t*.

- This simple trick which we discovered almost **50 years after the Black-Merton-Scholes formulas** in our 2022 JF paper, has revolutionary implications for risk and return computations of contingent claims.

How to Compute Variance and Covariance of Contingent Claim Returns

The $\mathbb{R}$ measure also allows an **analytical solution of the second moment and co-moment of contingent claims at a finite horizon H.** For example,

$$\mathrm{E}_t\left[C_H^2\right]$$

$$\mathrm{E}_t\left[C_{1H} \cdot C_{2H}\right]$$

= This is what we do in the next paper

How to Compute Portfolio Return and Variance

The $\mathbb{R}$ measure allows an analytical solutions of the **expected returns and variance** of a portfolio consisting of arbitrary contingent claims at a finite horizon H.

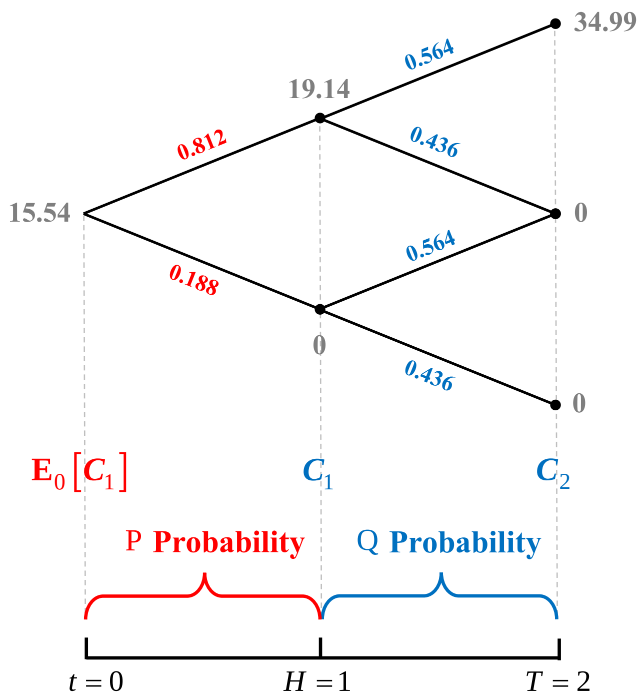
For example, given any contingent claim F^i with weights ω_i,

$$\mathbb{E}_t\left[\sum_{i=1}^{N}\omega_i F_H^i\right] = \sum_{i=1}^{N}\omega_i\,\mathbb{E}_t\left[F_H^i\right]$$

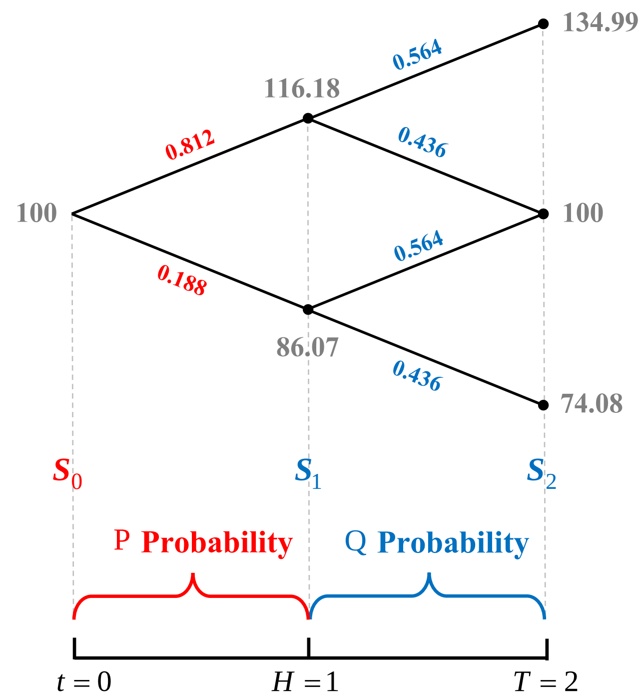
$$\text{Var}_t\left[\sum_{i=1}^{N}\omega_i F_H^i\right] = \sum_{i=1}^{N}\omega_i^2\text{Var}_t\left[F_H^i\right] + \sum_{i=1}^{N}\sum_{j\neq i}\omega_i\omega_j\text{Cov}_t\left[F_H^i, F_H^j\right]$$

The $\mathbb{R}$ Measure: A Binomial Tree Example

- The binomial tree for **expected call price** and **stock price** under the $\mathbb{R}$ measure:



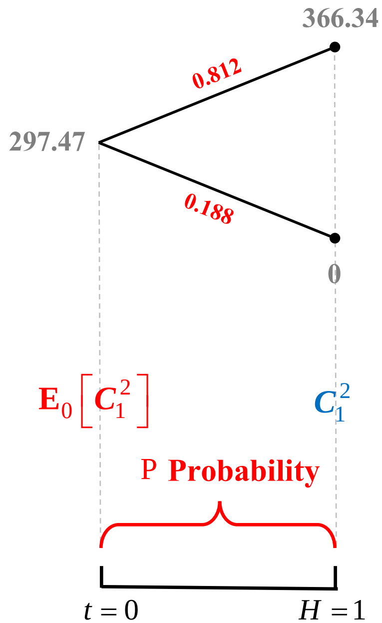
Expected Call Price



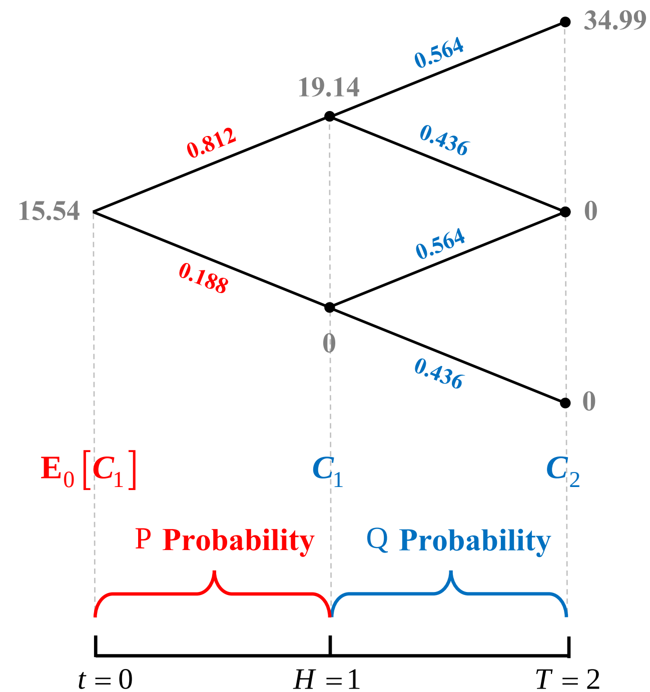
Stock Price

The ℝ Measure: A Binomial Tree Example

- The binomial tree for **expected call price squared** and **expected call price** under the ℝ measure:



Expected Call Price Squared



Expected Call Price

Applications of the $\mathbb{R}$ Measure

The Black-Scholes model

- The Black-Scholes model uses an equivalent probability measure $\mathbb{Q}$, whose discovery was made heuristically by Cox and Ross (1976) and formally by Harrison and Kreps (1979).

$$W_s^{\mathbb{Q}} = W_s^{\mathbb{P}} + \int_0^s \gamma \mathrm{d}u$$

- Define a probability measure $\mathbb{R}$ which is equivalent to $\mathbb{P}$, whose Brownian motion/Wiener process is given as follows:

$$W_s^{\mathbb{R}} = W_s^{\mathbb{P}} + \int_0^s \mathbf{1}_{\{u \geq H\}} \gamma \mathrm{d}u$$

Applications of the $\mathbb{R}$ Measure

The Black-Scholes model

- Under the physical measure $\mathbb{P}$, the underlying asset process S is given as

$$\frac{\mathrm{d}S_s}{S_s} = \mu \mathrm{d}s + \sigma \mathrm{d}W_s^{\mathbb{P}},$$

- By the Girsanov theorem, define $W_s^{\mathbb{Q}} = W_s^{\mathbb{P}} + \int_0^s \gamma \mathrm{d}u$, where $\gamma = (\mu - r)/\sigma$ is the market price of risk (MPR), the stock price under the risk neutral measure $\mathbb{Q}$ is given as

$$\frac{\mathrm{d}S_s}{S_s} = r \mathrm{d}s + \sigma \mathrm{d}W_s^{\mathbb{Q}}.$$

- Substituting W_S^{P} with $\mathrm{d}W_s^{\mathbb{P}} = \mathrm{d}W_s^{\mathbb{R}} - \mathbf{1}_{\{s>H\}}\gamma \mathrm{d}s$ gives the stock price process under the $\mathbb{R}$ measure as follows:

$$\begin{aligned}\frac{\mathrm{d}S_s}{S_s} &= \left(r + \gamma\sigma \mathbf{1}_{\{s<H\}}\right)\mathrm{d}s + \sigma \mathrm{d}W_s^{\mathbb{R}} \\ &= \left(\mu \mathbf{1}_{\{s<H\}} + r \mathbf{1}_{\{s\geq H\}}\right)\mathrm{d}s + \sigma \mathrm{d}W_s^{\mathbb{R}}.\end{aligned}$$

Applications of the $\mathbb{R}$ Measure

The Black-Scholes model

- In our JF paper, we show the expectation of any contingent claim, for example, a call option can be computed as follows:

$$\mathbb{E}_t\left[C_H\right] = \mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-\int_H^T r_u \mathrm{d}u} C_T\right]$$

- Therefore, the expected European option price holding until future time H with strike price K is

$$\begin{aligned}\mathbb{E}_t\left[C_H\right] &= \mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-r(T-H)}(S_T-K)^+\right] \\ &= S_t \mathrm{e}^{\mu(H-t)}\mathcal{N}\left(\hat{d}_1\right) - K\mathrm{e}^{-r(T-H)}\mathcal{N}\left(\hat{d}_2\right),\end{aligned}$$

$$\hat{d}_1 = \frac{\ln(S_t/K) + \mu(H-t) + r(T-H) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

$$\hat{d}_2 = \frac{\ln(S_t/K) + \mu(H-t) + r(T-H) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Applications of the $\mathbb{R}$ Measure

Dynamic term structure model: the CIR model

- Cox, Ingersoll and Ross (CIR, 1985) assume the instantaneous short rate process is given as follows:

$$\mathrm{d}r_s = \alpha_r \left(m_r - r_s\right) \mathrm{d}s + \sigma_r \sqrt{r_s} \mathrm{d}W_s^{\mathbb{P}}$$

- Assume the market price of risk and the Girsanov change between Brownian motions are given by $W_s^{\mathbb{Q}} = W_s^{\mathbb{P}} + \int_0^s \gamma_r \sqrt{r_u} \mathrm{d}u$
- Then the short rate process under the $\mathbb{R}$ measure can be derived as follows:

$$\mathrm{d}r_s = \left[\alpha_r \left(m_r - r_s\right) - \mathbf{1}_{\{s \geq H\}} \gamma_r \sigma_r r_s\right] \mathrm{d}s + \sigma_r \sqrt{r_s} \mathrm{d}W_s^{\mathbb{R}}$$

Applications of the $\mathbb{R}$ Measure

Dynamic term structure model: the CIR model

- The expected future price of a T-maturity pure discount bond at time H is given under the CIR model as follows. For all $t \leq H \leq T$,

$$\mathbb{E}_t^{\mathbb{P}}\left[P(H,T)\right] = \mathbb{E}_t^{\mathbb{R}}\left[\exp\left(-\int_H^T r_u \mathrm{d}u\right)\right]$$
$$= \exp\left(-A^{(0,1)}_{\alpha_r^*,m_r^*}(T-H) - A^{(b_1,0)}_{\alpha_r,m_r}(H-t) - B^{(b_1,0)}_{\alpha_r,m_r}(H-t)\cdot r_t\right),$$

where $b_1 = B^{(0,1)}_{\alpha_r^*,m_r^*}(T-H)$, $\alpha_r^* = \alpha_r + \gamma_r\sigma_r$, $m_r^* = \frac{\alpha_r m_r}{\alpha_r+\gamma_r\sigma_r}$ and for any well-behaved b, c, α and m, $A^{(b,c)}_{\alpha,m}(\tau)$ and $B^{(b,c)}_{\alpha,m}(\tau)$ are given by

$$A^{(b,c)}_{\alpha,m}(\tau) = -\frac{2\alpha m}{\sigma_r^2}\ln\left(\frac{2\beta \mathrm{e}^{\frac{1}{2}(\beta+\alpha)\tau}}{b\sigma_r^2\left(\mathrm{e}^{\beta\tau}-1\right)+\beta-\alpha+\mathrm{e}^{\beta\tau}(\beta+\alpha)}\right),$$

$$B^{(b,c)}_{\alpha,m}(\tau) = \frac{b\left(\beta+\alpha+\mathrm{e}^{\beta\tau}(\beta-\alpha)\right)+2c\left(\mathrm{e}^{\beta\tau}-1\right)}{b\sigma_r^2\left(\mathrm{e}^{\beta\tau}-1\right)+\beta-\alpha+\mathrm{e}^{\beta\tau}(\beta+\alpha)},$$

with $\beta = \sqrt{\alpha^2 + 2\sigma_r^2 c}$.

Applications of the $\mathbb{R}$ Measure

Expected future prices of contingent claims under the following models:

1. Equity option pricing models of Black and Scholes (1973), Cox and Ross (1976), Merton (1976), Hull and White (1987), and Rubinstein (1991).
2. Corporate debt pricing models of Merton (1974), Black and Cox (1976), and Leland and Toft (1996).
3. Term structure models of Dai and Singleton (2000, 2002), Ahn, Dittmar, and Gallant (2002), Leippold and Wu (2003), and Collin-Dufresne, Goldstein, and Jones (2008) for pricing default-free bonds.
4. Credit default swap pricing model of Longstaff, Mithal, and Neis (2005);
5. VIX futures and the variance swaps models of Dew-Becker et al. (2017), Eraker and Wu (2017), Johnson (2017) and Cheng (2019).
6. Currency option model of Garman and Kohlhagen (1983).
7. Various Fourier transform-based contingent claim models of Heston (1993), Duffie, Pan, and Singleton (2000), Carr et al. (2002), and others, summarized in Section II.C, among others.

The $\mathbb{R}_1^T$ Measure

- Another way for the computation of expected prices:

$$\mathbb{E}_t\left[C_H\right] = \mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-\int_H^T r_u \mathrm{d}u} C_T\right]$$
$$= \mathbb{E}_t^{\mathbb{R}}\left[P(H,T)\right] \cdot \mathbb{E}_t^{\mathbb{R}}\left[\frac{\mathrm{e}^{-\int_H^T r_u \mathrm{d}u}}{\mathbb{E}_t^{\mathbb{R}}\left[P(H,T)\right]} \cdot C_T\right]$$
$$= \mathbb{E}_t^{\mathbb{P}}\left[P(H,T)\right] \cdot \mathbb{E}_t^{\mathbb{R}_1^T}\left[C_T\right]$$

where

$$\mathbb{E}_t^{\mathbb{P}}\left[P(H,T)\right] = \mathbb{E}_t^{\mathbb{R}}\left[P(H,T)\right] = \mathbb{E}_t^{\mathbb{R}}\left[\mathbb{E}_H^{\mathbb{R}}\left[\mathrm{e}^{-\int_H^T r_u \mathrm{d}u}\right]\right] = \mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-\int_H^T r_u \mathrm{d}u}\right]$$

and for any random variable Z_T,

$$\mathbb{E}_t^{\mathbb{R}_1^T}\left[Z_T\right] = \mathbb{E}_t^{\mathbb{R}}\left[\frac{\mathrm{e}^{-\int_H^T r_u \mathrm{d}u}}{\mathbb{E}_t^{\mathbb{R}}\left[P(H,T)\right]} \cdot Z_T\right]$$

Applications of the $\mathbb{R}_1^T$ Measure

The Merton model

- Assume the asset price S is described by

$$\frac{\mathrm{d}S_s}{S_s} = \mu(s)\mathrm{d}s + \sigma(s)\mathrm{d}W_{1s}^{\mathbb{P}},$$

- The price of a T-maturity pure discount bond $P(\cdot,T)$ follows as

$$\frac{\mathrm{d}P(s,T)}{P(s,T)} = \mu_P(s,T)\mathrm{d}s + \sigma_P(s,T)\mathrm{d}W_{2s}^{\mathbb{P}},$$

- Consider a European call option C written on the asset S with a strike price of K, and an option expiration date equal to T. Using the T-maturity pure discount bond as the numeraire, the expected price of this option is given by

$$\begin{aligned}\mathbb{E}_t\left[C_H\right] &= \mathbb{E}_t^{\mathbb{P}}\left[P(H,T)\right]\mathbb{E}_t^{\mathbb{R}_1^T}\left[(S_T-K)^+\right] \\ &= \mathbb{E}_t^{\mathbb{P}}\left[P(H,T)\right]\mathbb{E}_t^{\mathbb{R}_1^T}\left[\frac{(S_T-K)^+}{P(T,T)}\right] \\ &= \mathbb{E}_t^{\mathbb{P}}\left[P(H,T)\right]\mathbb{E}_t^{\mathbb{R}_1^T}\left[(V_T-K)^+\right],\end{aligned}$$

where $V=S/P(\cdot,T)$ is the asset price normalized by the numeraire.

Applications of the $\mathbb{R}_1^T$ Measure

The Merton model

- We require that the asset price normalized by the bond price numeraire be distributed lognormally under the $\mathbb{R}_1^T$ measure, so that the sufficient assumptions for this to occur are:
  - i) the physical drift of the asset price process is of the form $\mu(s) = r_s + \gamma(s)$, where the risk premium $\gamma(s)$ is deterministic;
  - ii) the short rate process r_s, and the bond price process $P(\cdot,T)$ are consistent with the various multifactor Gaussian term structure models (see Dai and Singleton, 2000; Heath et al., 1992).
- Under the above assumptions, the solution of the expected price of call option is

$$\mathbb{E}_t[C_H] = \mathbb{E}_t\left[S_H\right] \mathcal{N}\left(\hat{d}_1\right) - K\,\mathbb{E}_t\left[P(H,T)\right] \mathcal{N}\left(\hat{d}_2\right),$$

where

$$\hat{d}_1 = \frac{1}{v_p} \ln \frac{\mathbb{E}_t\left[S_H\right]}{\mathbb{E}_t\left[P(H,T)\right] K} + \frac{v_p}{2}, \qquad \hat{d}_2 = \frac{1}{v_p} \ln \frac{\mathbb{E}_t\left[S_H\right]}{\mathbb{E}_t\left[P(H,T)\right] K} - \frac{v_p}{2},$$

$$v_p = \sqrt{\int_t^T \left(\sigma(u)^2 - 2\rho(u)\sigma(u)\sigma_P(u,T) + \sigma_P(u,T)^2\right) \mathrm{d}u.}$$

Applications of the $\mathbb{R}_1^T$ Measure

The Collin-Dufresne and Goldstein model

- Collin-Dufresne and Goldstein (CDG, 2001) allow the issuing firm to continuously adjust its capital structure to maintain a stationary mean-reverting leverage ratio. The firm's asset return process and the short rate process under the CDG model are given as follows:

$$\frac{\mathrm{d}S_s}{S_s} = (r_s + \gamma^S \sigma)\mathrm{d}s + \sigma\left(\rho \mathrm{d}W_{1s}^{\mathbb{P}} + \sqrt{1-\rho^2}\mathrm{d}W_{2s}^{\mathbb{P}}\right),$$
$$\mathrm{d}r_s = \alpha_r\left(m_r - r_s\right)\mathrm{d}s + \sigma_r \mathrm{d}W_{1s}^{\mathbb{P}},$$

- The market prices of risks associated with the two Brownian motions are given

$$\gamma_{1s} = \gamma_r,$$
$$\gamma_{2s} = \frac{1}{\sqrt{1-\rho^2}}\left(-\rho\gamma_r + \gamma^S\right).$$

Applications of the $\mathbb{R}_1^T$ Measure

The Collin-Dufresne and Goldstein model

- CDG assume that the face value of firm's total debt F^D follows the following stationary mean reverting process:

$$\frac{\mathrm{d}F_s^D}{F_s^D} = \lambda \left(\ln S_s - \nu - \phi r_s - \ln F_s^D\right) \mathrm{d}s.$$

- Defining the log-leverage ratio as $l \triangleq \ln(F^D/S)$, and applying Itô's lemma, we obtain

$$\mathrm{d}l_s = \lambda \left(\bar{l} - \frac{\sigma\gamma^S}{\lambda} - \left(\frac{1}{\lambda} + \phi\right) r_s - l_s\right) \mathrm{d}s - \sigma \left(\rho \mathrm{d}W_{1s}^{\mathbb{P}} + \sqrt{1-\rho^2}\mathrm{d}W_{2s}^{\mathbb{P}}\right),$$

where $\bar{l} \triangleq \frac{\sigma^2}{2\lambda} - \nu.$

Applications of the $\mathbb{R}_1^T$ Measure

The Collin-Dufresne and Goldstein model

- Using Lemma 2 and Internet Appendix Section I.D in our JF paper, the processes of the state variables $\{l,r\}$ under the $\mathbb{R}_1^T$ measure are

$$\mathrm{d}l_s = \lambda\left(\bar{l} - \left(\frac{1}{\lambda}+\phi\right) r_s - l_s + \frac{\rho\sigma\sigma_r}{\lambda} B_{\alpha_r}(T-s) - \left(\frac{\rho\sigma\sigma_r}{\lambda} B_{\alpha_r}(H-s) + \frac{\sigma\gamma^S}{\lambda}\right)\mathbb{1}_{\{s<H\}}\right)\mathrm{d}s$$
$$- \sigma\left(\rho \mathrm{d}W_{1s}^{\mathbb{R}_1^T} + \sqrt{1-\rho^2}\mathrm{d}W_{2s}^{\mathbb{R}_1^T}\right),$$
$$\mathrm{d}r_s = \alpha_r\left(m_r - \frac{\sigma_r\gamma_r}{\alpha_r} - r_s - \frac{\sigma_r^2}{\alpha_r} B_{\alpha_r}^{(T-s)} + \left(\frac{\sigma_r^2}{\alpha_r} B_{\alpha_r}(H-s) + \frac{\sigma_r\gamma_r}{\alpha_r}\right)\mathbb{1}_{\{s<H\}}\right)\mathrm{d}s + \sigma_r \mathrm{d}W_{1s}^{\mathbb{R}_1^T},$$

where $B_\alpha(\tau) = (1/\alpha)(1-\mathrm{e}^{-\alpha\tau})$.

- Then the expected future price of a risky discount bond is given by

$$\mathbb{E}_t\left[D(H,T)\right] = \mathbb{E}_t^{\mathbb{P}}\left[P(H,T)\right]\left(1 - \omega\,\mathbb{E}_t^{\mathbb{R}_1^T}\left[\mathbf{1}_{\{\tau_{[t,T]}<T\}}\right]\right)$$

Applications of the $\mathbb{R}_1^T$ Measure

Expected future prices of contingent claims under the following models:

1. The stochastic interest-rate-based equity option pricing model of Merton (1973b) and its extensions.
2. The stochastic interest-rate-based corporate debt pricing models of Longstaff and Schwartz (1995), Jarrow, Lando, and Turnbull (1997), and Collin-Dufresne and Goldstein (2001).
3. Various term structure models for pricing bond options and caps, such as Dai and Singleton (2000, 2002), Collin-Dufresne, Goldstein, and Jones (2008); Ahn, Dittmar, and Gallant (2002), Leippold and Wu (2003); Heath, Jarrow, and Morton (1992), Miltersen, Sandmann, and Sondermann (1997), Brace, Ga̧tarek, and Musiela (1997), and Jamshidian (1997).
4. The currency option pricing models of Grabbe (1983), Amin and Jarrow (1991), and Hilliard, Madura, and Tucker (1991), among others.

The *R*-Transform

- Duffie et al. (2000) define a transform $\phi : \mathcal{C}^N \times \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{C}$ of Y_T conditional on $\mathcal{F}_t$, when well defined for all $t \leq T$, as

$$\phi^{Q}(z;t,T) \triangleq \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(Y_u)\mathrm{d}u\right)\exp\left(z'Y_T\right)\right]$$

- To extend the *Q*-transform and compute the expected prices of contingent claims, we define the *R*-transform $\phi^R : \mathcal{C}^N \times \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{C}$ of Y_T conditional on $\mathcal{F}_t$, when well defined for all $t \leq H \leq T$, as

$$\phi^{R}(z;t,T,H) \triangleq \mathbb{E}_t^{\mathbb{R}}\left[\exp\left(-\int_H^T r(Y_u)\mathrm{d}u\right)\exp\left(z'Y_T\right)\right]$$

Applications of the *R*-Transform

The SVJJ model

- Duffie, Pan and Singleton (2000) propose a stochastic volatility jump-based equity option (SVJJ) model, which assumes that the log asset price process, $Y = \ln S$, and the volatility factor follow the following processes under the physical measure $\mathbb{P}$:

$$\mathrm{d}Y_s = \left(r - q + \gamma^S v_s + \gamma^J - \frac{v_s}{2}\right)\mathrm{d}s + \sqrt{v_{s-}}\left(\rho \mathrm{d}W_{1s}^{\mathbb{P}} + \sqrt{1-\rho^2}\mathrm{d}W_{2s}^{\mathbb{P}}\right) + \mathrm{d}\left(\sum_{i=1}^{N_s} J_{S,i}\right) - \lambda\bar{\mu}\mathrm{d}s,$$

$$\mathrm{d}v_s = \alpha_v\left(m_v - v_s\right)\mathrm{d}s + \sigma_v\sqrt{v_{s-}}\mathrm{d}W_{1s}^{\mathbb{P}} + \mathrm{d}\left(\sum_{i=1}^{N_s} J_{v,i}\right),$$

- The risk-neutral process under risk-neutral measure $\mathbb{Q}$ is given as:

$$\mathrm{d}Y_s = \left(r - q - \frac{v_s}{2}\right)\mathrm{d}s + \sqrt{v_{s-}}\left(\rho \mathrm{d}W_{1s}^{\mathbb{Q}} + \sqrt{1-\rho^2}\mathrm{d}W_{2s}^{\mathbb{Q}}\right) + \mathrm{d}\left(\sum_{i=1}^{N_s} J_{S,i}\right) - \lambda^*\bar{\mu}^*\mathrm{d}s,$$

$$\mathrm{d}v_s = \left[\alpha_v\left(m_v - v_s\right) - \gamma^v v_s\right]\mathrm{d}s + \sigma_v\sqrt{v_{s-}}\mathrm{d}W_{1s}^{\mathbb{Q}} + \mathrm{d}\left(\sum_{i=1}^{N_s} J_{v,i}\right).$$

Applications of the *R*-Transform

The SVJJ model

- Under the $\mathbb{R}$ measure, the dynamics of log asset price process and the volatility factor process are derived as follows:

$$\mathrm{d}Y_s = \left(r - q - \frac{v_s}{2} + \mathbf{1}_{\{s<H\}}\gamma^S v_s + \mathbf{1}_{\{s<H\}}\gamma^J\right)\mathrm{d}s + \sqrt{v_{s-}}\left(\rho \mathrm{d}W_{1s}^{\mathbb{R}} + \sqrt{1-\rho^2}\mathrm{d}W_{2s}^{\mathbb{R}}\right)$$
$$+ \mathrm{d}\left(\sum_{i=1}^{N_s} J_{S,i}\right) - \mathbf{1}_{\{s<H\}}\lambda\bar{\mu}\mathrm{d}s - \mathbf{1}_{\{s\geq H\}}\lambda^*\bar{\mu}^*\mathrm{d}s,$$

$$\mathrm{d}v_s = \left[\alpha_v(m_v - v_s) - \mathbf{1}_{\{s\geq H\}}\gamma^v v_s\right]\mathrm{d}s + \sigma_v\sqrt{v_{s-}}\mathrm{d}W_{1s}^{\mathbb{R}} + \mathrm{d}\left(\sum_{i=1}^{N_s} J_{v,i}\right).$$

- The *R*-transform of the SVJJ model defined as below can be used to obtain the expected future prices of European call and put options:

$$\phi^R(z) \triangleq \mathbb{E}_t^{\mathbb{R}}\left[\exp\left(-\int_H^T r\mathrm{d}u + zY_T\right)\right]$$
$$= \exp\left(-r(T-H) + A_1^{(z_1^*,z_2^*)}(T-H) + A_2^{(z_1,z_2)}(H-t) + B_2^{(z_1,z_2)}(H-t)\cdot v_t + z\cdot Y_t\right)$$

How to Compute Higher Moment

- Why **does** the $\mathbb{R}$ measure work for the **expected price**?

$$
\begin{aligned}
\mathbb{E}_t\left[C_H\right] &= \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\right]\right] \\
&= \mathbb{E}_t^{\mathbb{R}}\left[\mathbb{E}_H^{\mathbb{R}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\right]\right] \\
&= \mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\right]
\end{aligned}
$$

- Why **doesn't** the $\mathbb{R}$ measure work for the **higher moment**?

$$
\begin{aligned}
\mathbb{E}_t\left[C_H^2\right] &= \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(S_T-K)^+\right]\right] \\
&= \mathbb{E}_t^{\mathbb{R}}\left[\mathbb{E}_H^{\mathbb{R}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\right]\cdot\mathbb{E}_H^{\mathbb{R}}\left[\mathrm{e}^{-r(T-H)}(S_T-K)^+\right]\right] \\
&\neq \mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\cdot\mathrm{e}^{-r(T-H)}(S_T-K)^+\right]
\end{aligned}
$$

The Parallel Process

- Introduce **a parallel process** which has **the exact same process as the underlying process before *H***, and has an **independent and identically-distributed process after time *H*.**
- For example, under the Black-Scholes model, the underlying asset price process S and the corresponding parallel process $\dot{S}$ are given as follows:
    - Assume $S_0 = \dot{S}_0$, then we have under $\mathbb{P}$:

$$\frac{\mathrm{d}S_s}{S_s} = \mu \mathrm{d}s + \sigma \mathrm{d}W_s^{\mathbb{P}},$$
$$\frac{\mathrm{d}\dot{S}_s}{\dot{S}_s} = \mu \mathrm{d}s + \sigma \mathrm{d}\dot{W}_s^{\mathbb{P}},$$

where

$$\mathrm{d}\dot{W}_s^{\mathbb{P}} = \mathbf{1}_{\{s<H\}}\mathrm{d}W_s^{\mathbb{P}} + \sigma \mathbf{1}_{\{s \geq H\}}\mathrm{d}\bar{W}_s^{\mathbb{P}}$$

with $W^{\mathbb{P}}$ and $\bar{W}^{\mathbb{P}}$ being two independent Brownian motions.

The Parallel Process

- Under $\mathbb{Q}$:

$$\frac{\mathrm{d}S_s}{S_s} = r\mathrm{d}s + \sigma \mathrm{d}W_s^{\mathbb{Q}},$$

$$\frac{\mathrm{d}\dot{S}_s}{\dot{S}_s} = r\mathrm{d}s + \sigma \mathrm{d}\dot{W}_s^{\mathbb{Q}},$$

where

$$\mathrm{d}\dot{W}_s^{\mathbb{P}} = \mathbf{1}_{\{s<H\}}\mathrm{d}W_s^{\mathbb{Q}} + \sigma\mathbf{1}_{\{s\geq H\}}\mathrm{d}\bar{W}_s^{\mathbb{Q}}$$

with $W^{\mathbb{Q}}$ and $\bar{W}^{\mathbb{Q}}$ being two independent Brownian motions.

- Under $\mathbb{R}$:

$$\frac{\mathrm{d}S_s}{S_s} = \left(\mu\mathbf{1}_{\{s<H\}} + r\mathbf{1}_{\{s\geq H\}}\right)\mathrm{d}s + \sigma \mathrm{d}W_s^{\mathbb{R}},$$

$$\frac{\mathrm{d}\dot{S}_s}{\dot{S}_s} = \left(\mu\mathbf{1}_{\{s<H\}} + r\mathbf{1}_{\{s\geq H\}}\right)\mathrm{d}s + \sigma \mathrm{d}\dot{W}_s^{\mathbb{R}},$$

where

$$\mathrm{d}\dot{W}_s^{\mathbb{R}} = \mathbf{1}_{\{s<H\}}\mathrm{d}W_s^{\mathbb{R}} + \sigma\mathbf{1}_{\{s\geq H\}}\mathrm{d}\bar{W}_s^{\mathbb{R}}$$

with $W^{\mathbb{R}}$ and $\bar{W}^{\mathbb{R}}$ being two independent Brownian motions.

The Parallel Process

- The expected second moment of holding a call option until future time H can be obtained as follows:

$$
\begin{aligned}
\mathbb{E}_t\left[C_H^2\right] &= \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(S_T-K)^+\right]\right] \\
&= \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(\dot{S}_T-K)^+\right]\right] \\
&= \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\cdot\mathrm{e}^{-r(T-H)}(\dot{S}_T-K)^+\right]\right] \\
&= \mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-2r(T-H)}(S_T-K)(\dot{S}_T-K)\mathbf{1}_{\{S_T>K,\dot{S}_T>K\}}\right] \\
&= \int_K^\infty\int_K^\infty \mathrm{e}^{-2r(T-H)}(S_T-K)(\dot{S}_T-K)p_t^{\mathbb{R}}\left(S_T,\dot{S}_T\right)\mathrm{d}S_T\mathrm{d}\dot{S}_T
\end{aligned}
$$

The Parallel Process

- The expected second moment of holding two call options based on two stocks until future time H can be obtained as follows:

$$\begin{aligned}
\mathbb{E}_t\left[C_{H,T_1}^{S_1,K_1}\cdot C_{H,T_2}^{S_2,K_2}\right] &= \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_{1T_1}-K_1\right)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(S_{2T_2}-K_2)^+\right]\right]\\
&= \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_{1T_1}-K_1\right)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(\dot{S}_{2T_2}-K_2)^+\right]\right]\\
&= \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_{1T_1}-K_1\right)^+\cdot\mathrm{e}^{-r(T-H)}(\dot{S}_{2T_2}-K_2)^+\right]\right]\\
&= \mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-2r(T-H)}(S_{1T_1}-K)(\dot{S}_{2T_2}-K)\mathbf{1}_{\{S_{1T_1}>K_1,\dot{S}_{2T_2}>K_2\}}\right]\\
&= \int_{K_1}^{\infty}\int_{K_2}^{\infty}\mathrm{e}^{-2r(T-H)}(S_{1T_1}-K)(\dot{S}_{2T_2}-K)p_t^{\mathbb{R}}\left(S_{1T_1},\dot{S}_{2T_2}\right)\mathrm{d}S_{1T_1}\,\mathrm{d}\dot{S}_{2T_2}
\end{aligned}$$

The Parallel Process

- The expected third moment of holding a call option until future time H can be obtained as follows:

$$\begin{aligned}
\mathbb{E}_t\left[C_H^3\right] &= \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(S_T-K)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(S_T-K)^+\right]\right] \\
&= \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(\dot{S}_T-K)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(\ddot{S}_T-K)^+\right]\right] \\
&= \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_T-K\right)^+\cdot\mathrm{e}^{-r(T-H)}(\dot{S}_T-K)^+\cdot\mathrm{e}^{-r(T-H)}(\ddot{S}_T-K)^+\right]\right] \\
&= \mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-3r(T-H)}(S_T-K)(\dot{S}_T-K)(\ddot{S}_T-K)^+\mathbf{1}_{\{S_T>K,\dot{S}_T>K,\ddot{S}_T>K\}}\right] \\
&= \int_K^\infty\int_K^\infty\int_K^\infty \mathrm{e}^{-3r(T-H)}(S_T-K)(\dot{S}_T-K)(\ddot{S}_T-K)p_t^{\mathbb{R}}\left(S_T,\dot{S}_T,\ddot{S}_T\right)\mathrm{d}S_T\mathrm{d}\dot{S}_T\mathrm{d}\ddot{S}_T
\end{aligned}$$

The Parallel Process

- The expected third moment of holding three call options based on three stocks until future time H can be obtained as follows:

$$
\begin{aligned}
&\mathbb{E}_t\left[C_{H,T_1}^{S_1,K_1}\cdot C_{H,T_2}^{S_2,K_2}\cdot C_{H,T_3}^{S_3,K_3}\right]\\
&=\mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_{1T_1}-K_1\right)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(S_{2T_2}-K_2)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(S_{3T_3}-K_3)^+\right]\right]\\
&=\mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_{1T_1}-K_1\right)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(\dot{S}_{2T_2}-K_2)^+\right]\cdot\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}(\ddot{S}_{3T_3}-K_3)^+\right]\right]\\
&=\mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_H^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-H)}\left(S_{1T_1}-K_1\right)^+\cdot\mathrm{e}^{-r(T-H)}(\dot{S}_{2T_2}-K)^+\cdot\mathrm{e}^{-r(T-H)}(\ddot{S}_{3T_3}-K_3)^+\right]\right]\\
&=\mathbb{E}_t^{\mathbb{R}}\left[\mathrm{e}^{-3r(T-H)}(S_{1T_1}-K_1)(\dot{S}_{2T_2}-K_2)(\ddot{S}_{3T_3}-K_3)^+\mathbf{1}_{\{S_{1T_1}>K_1,\dot{S}_{2T_2}>K_2,\ddot{S}_{3T_3}>K_3\}}\right]\\
&=\int_{K_1}^{\infty}\int_{K_2}^{\infty}\int_{K_3}^{\infty}\mathrm{e}^{-3r(T-H)}(S_{1T_1}-K_1)(\dot{S}_{2T_2}-K_2)(\ddot{S}_{3T_3}-K_3)p_t^{\mathbb{R}}\left(S_{1T_1},\dot{S}_{2T_2},\ddot{S}_{3T_3}\right)\mathrm{d}S_{1T_1}\mathrm{d}\dot{S}_{2T_2}\mathrm{d}\ddot{S}_{3T_3}
\end{aligned}
$$

Applications of the Parallel Process

The Black-Scholes model: the variance

- By deriving the joint distribution of S_T and $\dot{S}_T$ under the $\mathbb{R}$ measure, the second moment of option price holding until future time H with the strike price K under the Black-Scholes model is:

$$\mathbb{E}_t\left[C_H^2\right] = S_t^2 \mathrm{e}^{(2\mu+\sigma^2)(H-t)} \mathcal{N}\left(\hat{d}_3, \hat{d}_3\right) - 2KS_t\mathrm{e}^{\mu(H-t)-r(T-H)} \mathcal{N}\left(\hat{d}_1, \hat{d}_4\right) + K^2 \mathrm{e}^{-2r(T-H)} \mathcal{N}\left(\hat{d}_2, \hat{d}_2\right),$$

where $\mathcal{N}(x, y) = F(X < x, Y < y)$, with $\binom{X}{Y} \sim N\left[\binom{0}{0}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]$ and $\rho = (H-t)/(T-t)$

$$\hat{d}_1 = \frac{\ln(S_t/K) + \mu(H-t) + r(T-H) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

$$\hat{d}_2 = \frac{\ln(S_t/K) + \mu(H-t) + r(T-H) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

$$\hat{d}_3 = \frac{\ln(S_t/K) + (\mu+\sigma^2)(H-t) + r(T-H) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

$$\hat{d}_4 = \frac{\ln(S_t/K) + (\mu+\sigma^2)(H-t) + r(T-H) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

- The variance: $\mathrm{Var}_t[C_H] = \mathbb{E}_t\left[C_H^2\right] - \mathbb{E}_t^2\left[C_H\right]$

Applications of the Parallel Process

The Black-Scholes model: the covariance

- By deriving the joint distribution of S_{1T_1} and $\dot{S}_{2T_2}$ under the $\mathbb{R}$ measure, the covariance of two options holding until future time H under the Black-Scholes model is:

$$\mathbb{E}_t\left[C^{S_1,K_1}_{H,T_1} C^{S_2,K_2}_{H,T_2}\right] = S_{1t}S_{2t}\mathrm{e}^{(\mu_1+\mu_2+\rho\sigma_1\sigma_2)(H-t)}\mathcal{N}\left(\hat{d}_{31},\hat{d}_{32}\right) - K_1 S_{2t}\mathrm{e}^{\mu_2(H-t)-r(T_1-H)}\mathcal{N}\left(\hat{d}_{41},\hat{d}_{12}\right)$$
$$- K_2 S_{1t}\mathrm{e}^{\mu_1(H-t)-r(T_2-H)}\mathcal{N}\left(\hat{d}_{11},\hat{d}_{42}\right) + K_1K_2\mathrm{e}^{-r(T_1+T_2-2H)}\mathcal{N}\left(\hat{d}_{21},\hat{d}_{22}\right),$$

where $\mathcal{N}(x_1,x_2) = P(X_1 < x_1, X_2 < x_2)$, with $\binom{X_1}{X_2} \sim N\left[\binom{0}{0}, \begin{pmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{pmatrix}\right]$ and

$\rho_{12} = \rho(H-t)/\sqrt{(T_1-t)(T_2-t)}$, with

$$\hat{d}_{11} = \frac{\ln(S_{1t}/K_1) + \mu_1(H-t) + r(T_1-H) + \frac{1}{2}\sigma_1^2(T_1-t)}{\sigma_1\sqrt{T_1-t}}, \quad \hat{d}_{31} = \frac{\ln(S_{1t}/K_1) + (\mu_1+\rho\sigma_1\sigma_2)(H-t) + r(T_1-H) + \frac{1}{2}\sigma_1^2(T_1-t)}{\sigma_1\sqrt{T_1-t}},$$
$$\hat{d}_{12} = \frac{\ln(S_{2t}/K_2) + \mu_2(H-t) + r(T_2-H) + \frac{1}{2}\sigma_2^2(T_2-t)}{\sigma_2\sqrt{T_2-t}}, \quad \hat{d}_{32} = \frac{\ln(S_{2t}/K_2) + (\mu_2+\rho\sigma_1\sigma_2)(H-t) + r(T_2-H) + \frac{1}{2}\sigma_2^2(T_2-t)}{\sigma_2\sqrt{T_2-t}},$$
$$\hat{d}_{21} = \frac{\ln(S_{1t}/K_1) + \mu_1(H-t) + r(T_1-H) - \frac{1}{2}\sigma_1^2(T_1-t)}{\sigma_1\sqrt{T_1-t}}, \quad \hat{d}_{41} = \frac{\ln(S_{1t}/K_1) + (\mu_1+\rho\sigma_1\sigma_2)(H-t) + r(T_1-H) - \frac{1}{2}\sigma_1^2(T_1-t)}{\sigma_1\sqrt{T_1-t}},$$
$$\hat{d}_{22} = \frac{\ln(S_{2t}/K_2) + \mu_2(H-t) + r(T_2-H) - \frac{1}{2}\sigma_2^2(T_2-t)}{\sigma_2\sqrt{T_2-t}}, \quad \hat{d}_{42} = \frac{\ln(S_{2t}/K_2) + (\mu_2+\rho\sigma_1\sigma_2)(H-t) + r(T_2-H) - \frac{1}{2}\sigma_2^2(T_2-t)}{\sigma_2\sqrt{T_2-t}}.$$

- The covariance: $\mathrm{Cov}_t\left[C^{S_1,K_1}_{H,T_1}, C^{S_2,K_2}_{H,T_2}\right] = \mathbb{E}_t\left[C^{S_1,K_1}_{H,T_1} C^{S_2,K_2}_{H,T_2}\right] - \mathbb{E}_t\left[C^{S_1,K_1}_{H,T_1}\right]\mathbb{E}_t\left[C^{S_2,K_2}_{H,T_2}\right]$

Applications of the Parallel Processes

The Black-Scholes model: the skewness

- By deriving the joint distribution of S_T, $\dot{S}_T$ and $\ddot{S}_T$ under the $\mathbb{R}$ measure, the third moment of option price holding until future time H with the strike price K under the Black-Scholes model is:

$$\mathbb{E}_t\left[C_H^3\right] = S_t^3 \mathrm{e}^{(3\mu+3\sigma^2)(H-t)} \mathcal{N}\left(\hat{d}_5, \hat{d}_5, \hat{d}_5\right) - 3KS_t^2 \mathrm{e}^{(2\mu+\sigma^2)(H-t)-r(T-H)} \mathcal{N}\left(\hat{d}_3, \hat{d}_3, \hat{d}_6\right)$$
$$+ 3K^2 S_t \mathrm{e}^{\mu(H-t)-2r(T-H)} \mathcal{N}\left(\hat{d}_1, \hat{d}_4, \hat{d}_4\right) - K^3 \mathrm{e}^{-2r(T-H)} \mathcal{N}\left(\hat{d}_2, \hat{d}_2, \hat{d}_2\right),$$

where $\mathcal{N}(x, y, z) = F(X < x, Y < y, Z < z)$, with $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}\right]$, $\rho = (H-t)/(T-t)$

$$\hat{d}_5 = \frac{\ln(S_t/K) + (\mu + 2\sigma^2)(H-t) + r(T-H) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$
$$\hat{d}_6 = \frac{\ln(S_t/K) + (\mu + 2\sigma^2)(H-t) + r(T-H) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

- The skewness: $\text{Skew}_t[C_H] = \dfrac{\mathbb{E}_t\left[C_H^3\right] - 3\,\mathbb{E}_t\left[C_H\right]\mathbb{E}_t\left[C_H^2\right] + 2\,\mathbb{E}_t^3\left[C_H\right]}{\left(\mathbb{E}_t\left[C_H^2\right] - \mathbb{E}_t^2\left[C_H\right]\right)^{3/2}}$

Applications of the Parallel Process

Dynamic term structure model: the CIR model

- Under the CIR model, the expected second moment of a T-maturity pure discount bond at time H is given as follows: For all $t \leq H \leq T$,

$$\mathbb{E}_t\left[P(H,T)^2\right] = \mathbb{E}_t^{\mathbb{R}}\left[\exp\left(-\int_H^T r_u \mathrm{d}u\right)\cdot\exp\left(-\int_H^T \dot{r}_u \mathrm{d}u\right)\right]$$
$$= \exp\left(-2A^{(b^*,c^*)}_{\alpha_r^*,m_r^*}(T-H) - A^{(b,c)}_{\alpha_r,m_r}(H-t) - B^{(b,c)}_{\alpha_r,m_r}(H-t)\cdot r_t\right),$$

where $b^* = 0,\ c^* = 1,\ b = 2B^{(b^*,c^*)}_{\alpha_r^*,m_r^*}(T-H),\ c = 0$, and for any well-behaved b, c, α and m, $A^{(b,c)}_{\alpha,m}(\tau)$ and $B^{(b,c)}_{\alpha,m}(\tau)$ are given by

$$A^{(b,c)}_{\alpha,m}(\tau) = -\frac{2\alpha m}{\sigma_r^2}\ln\left(\frac{2\beta \mathrm{e}^{\frac{1}{2}(\beta+\alpha)\tau}}{b\sigma_r^2\left(\mathrm{e}^{\beta\tau}-1\right)+\beta-\alpha+\mathrm{e}^{\beta\tau}(\beta+\alpha)}\right),$$

$$B^{(b,c)}_{\alpha,m}(\tau) = \frac{b\left(\beta+\alpha+\mathrm{e}^{\beta\tau}(\beta-\alpha)\right)+2c\left(\mathrm{e}^{\beta\tau}-1\right)}{b\sigma_r^2\left(\mathrm{e}^{\beta\tau}-1\right)+\beta-\alpha+\mathrm{e}^{\beta\tau}(\beta+\alpha)},$$

with $\beta = \sqrt{\alpha^2 + 2\sigma_r^2 c}$.

Applications of the Parallel Process

The SVJJ model

- Under the $\mathbb{R}$ measure, the dynamics of the parallel log asset price process and the parallel volatility factor process are derived as follows:

$$\begin{aligned}\mathrm{d}\dot{Y}_s &= \left(r - q - \frac{\dot{v}_s}{2} + \mathbf{1}_{\{s<H\}}\gamma^S \dot{v}_s + \mathbf{1}_{\{s<H\}}\gamma^J\right)\mathrm{d}s \\ &+ \mathbf{1}_{\{s<H\}}\sqrt{\dot{v}_{s-}}\left(\rho \mathrm{d}W_{1s}^{\mathbb{R}} + \sqrt{1-\rho^2}\mathrm{d}W_{2s}^{\mathbb{R}}\right) + \mathbf{1}_{\{s\geq H\}}\sqrt{\dot{v}_{s-}}\left(\rho \mathrm{d}\dot{W}_{1s}^{\mathbb{R}} + \sqrt{1-\rho^2}\mathrm{d}\dot{W}_{2s}^{\mathbb{R}}\right) \\ &+ \mathrm{d}\left(\sum_{i=1}^{\dot{N}_s} \dot{J}_{\dot{S},i}\right) - \mathbf{1}_{\{s<H\}}\lambda\bar{\mu}\mathrm{d}s - \mathbf{1}_{\{s\geq H\}}\lambda^*\bar{\mu}^*\mathrm{d}s,\end{aligned}$$

$$\mathrm{d}\dot{v}_s = \left[\alpha_v(m_v - \dot{v}_s) - \mathbf{1}_{\{s\geq H\}}\gamma^v \dot{v}_s\right]\mathrm{d}s + \mathbf{1}_{\{s<H\}}\sigma_v\sqrt{\dot{v}_{s-}}\mathrm{d}W_{1s}^{\mathbb{R}} + \mathbf{1}_{\{s\geq H\}}\sigma_v\sqrt{\dot{v}_{s-}}\mathrm{d}\dot{W}_{1s}^{\mathbb{R}} + \mathrm{d}\left(\sum_{i=1}^{\dot{N}_s}\dot{J}_{\dot{v},i}\right).$$

- The extended *R*-transform of the SVJJ model defined as below can be used to obtain the expected second moment of European options:

$$\begin{aligned}\phi_2^R(z,\dot{z}) &\triangleq \mathbb{E}_t^{\mathbb{R}}\left[\exp\left(-\int_H^T 2r\mathrm{d}u + zY_T + \dot{z}\dot{Y}_T\right)\right] \\ &= \exp\Big(-2r(T-H) + A_1^{(z_1^*,z_2^*)}(T-H) + A_1^{(\dot{z}_1^*,\dot{z}_2^*)}(T-H) \\ &\quad + A_2^{(z_1,z_2)}(H-t) + B_2^{(z_1,z_2)}(H-t)\cdot v_t + (z+\dot{z})\cdot Y_t\Big).\end{aligned}$$

The $\mathbb{R}_2^T$ Measure

- A way for the computation of expected second moment:

$$
\begin{aligned}
\mathbb{E}_t\left[C_H^2\right] &= \mathbb{E}_t^{\mathbb{R}}\left[\mathbb{E}_H^{\mathbb{R}}\left[e^{-\int_H^T r_u du} C_T\right] \cdot \mathbb{E}_H^{\mathbb{R}}\left[e^{-\int_H^T r_u du} C_T\right]\right] \\
&= \mathbb{E}_t^{\mathbb{R}}\left[\mathbb{E}_H^{\mathbb{R}}\left[e^{-\int_H^T r_u du} C_T\right] \cdot \mathbb{E}_H^{\mathbb{R}}\left[e^{-\int_H^T \dot{r}_u du} \dot{C}_T\right]\right] \\
&= \mathbb{E}_t^{\mathbb{R}}\left[\mathbb{E}_H^{\mathbb{R}}\left[e^{-\int_H^T r_u du} C_T \cdot e^{-\int_H^T \dot{r}_u du} \dot{C}_T\right]\right] \\
&= \mathbb{E}_t^{\mathbb{R}}\left[e^{-\int_H^T r_u du} C_T \cdot e^{-\int_H^T \dot{r}_u du} \dot{C}_T\right] \\
&= \mathbb{E}_t^{\mathbb{R}}\left[P(H,T)^2\right] \cdot \mathbb{E}_t^{\mathbb{R}}\left[\frac{e^{-\int_H^T r_u du} e^{-\int_H^T \dot{r}_u du}}{\mathbb{E}_t^{\mathbb{R}}\left[P(H,T)^2\right]} \cdot C_T \dot{C}_T\right] \\
&= \mathbb{E}_t^{\mathbb{P}}\left[P(H,T)^2\right] \cdot \mathbb{E}_t^{\mathbb{R}_2^T}\left[C_T \dot{C}_T\right]
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{P}}\left[P(H,T)^2\right] &= \mathbb{E}_t^{\mathbb{R}}\left[P(H,T)^2\right] = \mathbb{E}_t^{\mathbb{R}}\left[\mathbb{E}_H^{\mathbb{R}}\left[e^{-\int_H^T r_u du}\right] \cdot \mathbb{E}_H^{\mathbb{R}}\left[e^{-\int_H^T r_u du}\right]\right] \\
&= \mathbb{E}_t^{\mathbb{R}}\left[\mathbb{E}_H^{\mathbb{R}}\left[e^{-\int_H^T r_u du}\right] \cdot \mathbb{E}_H^{\mathbb{R}}\left[e^{-\int_H^T \dot{r}_u du}\right]\right] = \mathbb{E}_t^{\mathbb{R}}\left[e^{-\int_H^T r_u du} \cdot e^{-\int_H^T \dot{r}_u du}\right]
\end{aligned}
$$

and or any random variable Z_T, $\mathbb{E}_t^{\mathbb{R}_2^T}\left[Z_T\right] = \mathbb{E}_t^{\mathbb{R}}\left[\frac{e^{-\int_H^T r_u du} \cdot e^{-\int_H^T \dot{r}_u du}}{\mathbb{E}_t^{\mathbb{R}}\left[P(H,T)^2\right]} \cdot Z_T\right]$

Applications of the Parallel Process

The Collin-Dufresne and Goldstein model

- For the CDG model, the parallel processes of the state variables $\{\dot{l}, \dot{r}\}$ under the $\mathbb{R}_2^T$ measure are given as

$$
\begin{aligned}
\mathrm{d}\dot{l}_s =& \lambda \left[\bar{l} - \left(\frac{1}{\lambda} + \phi \right) \dot{r}_s - \dot{l}_s + \frac{\rho\sigma\sigma_r}{\lambda} B_{\alpha_r}(T-s) \right. \\
& \left. - \left(\frac{\rho\sigma\sigma_r}{\lambda} \left(2B_{\alpha_r}(H-s) - B_{\alpha_r}(T-s) \right) + \frac{\sigma\gamma^S}{\lambda} \right) \mathbf{1}_{\{s<H\}} \right] \mathrm{d}s \\
& - \sigma \mathbf{1}_{\{s<H\}} \left(\rho \mathrm{d}W_{1s}^{\mathbb{R}_2^T} + \sqrt{1-\rho^2} \mathrm{d}W_{2s}^{\mathbb{R}_2^T} \right) - \sigma \mathbf{1}_{\{s\geq H\}} \left(\rho \mathrm{d}\dot{W}_{1s}^{\mathbb{R}_2^T} + \sqrt{1-\rho^2} \mathrm{d}\dot{W}_{2s}^{\mathbb{R}_2^T} \right), \\
\mathrm{d}\dot{r}_s =& \alpha_r \left[m_r - \frac{\sigma_r\gamma_r}{\alpha_r} - \dot{r}_s - \frac{\sigma_r^2}{\alpha_r} B_{\alpha_r}(T-s) + \left(\frac{\sigma_r^2}{\alpha_r} \left(2B_{\alpha_r}(H-s) - B_{\alpha_r}(T-s) \right) + \frac{\sigma_r\gamma_r}{\alpha_r} \right) \mathbf{1}_{\{s<H\}} \right] \\
& + \sigma_r \mathbf{1}_{\{s<H\}} \mathrm{d}W_{1s}^{\mathbb{R}_2^T} + \sigma_r \mathbf{1}_{\{s\geq H\}} \mathrm{d}\dot{W}_{1s}^{\mathbb{R}_2^T}.
\end{aligned}
$$

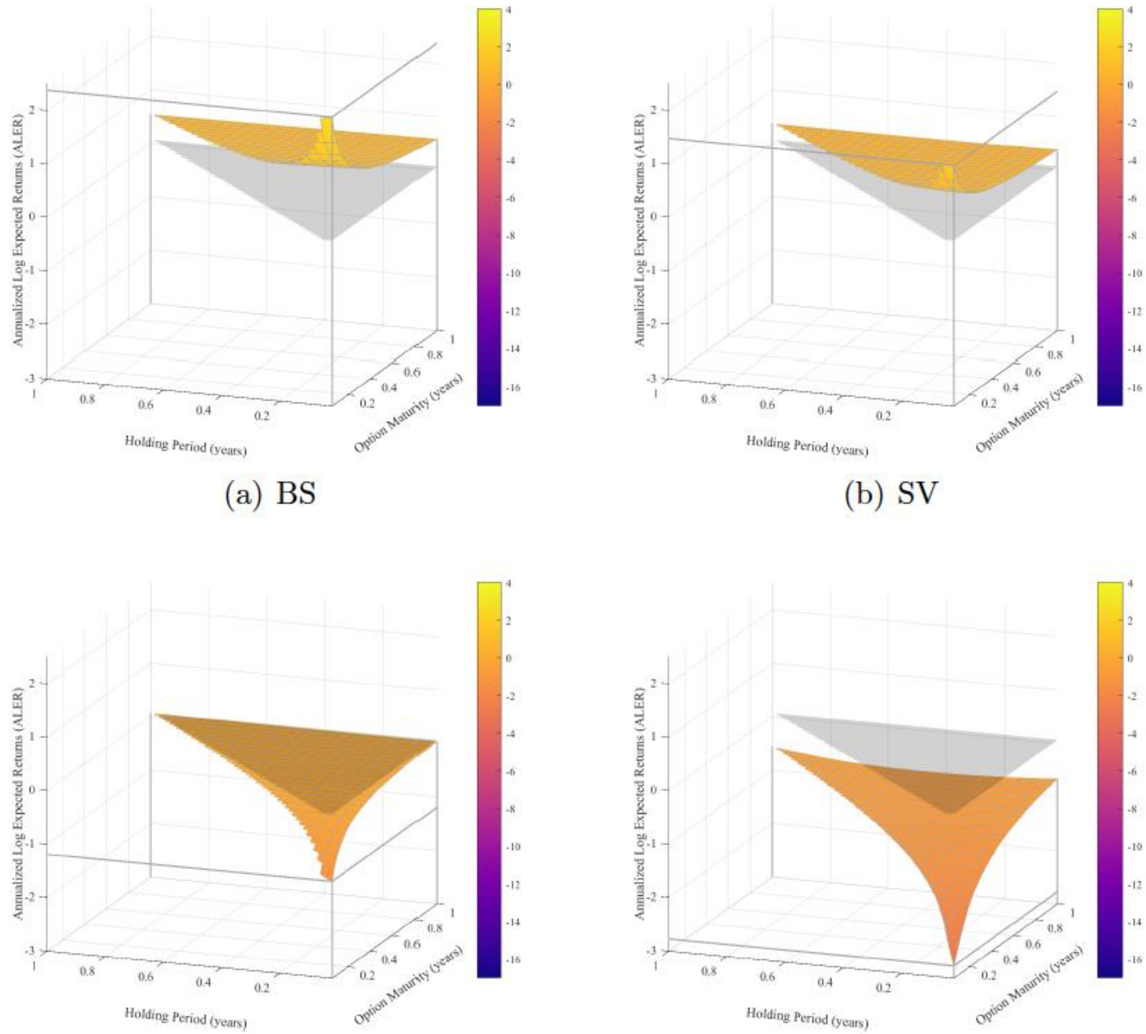
where $B_\alpha(\tau) = (1/\alpha)(1 - \mathrm{e}^{-\alpha\tau})$.

- Then the expected second moment of a risky discount bond is

$$
\begin{aligned}
\mathbb{E}_t \left[D(H,T)^2 \right] &= \mathbb{E}_t^{\mathbb{P}} \left[P(H,T)^2 \right] \mathbb{E}_t^{\mathbb{R}_2^T} \left[\left(1 - \omega \mathbf{1}_{\{\tau_{[t,T]}<T\}} \right) \left(1 - \omega \mathbf{1}_{\{\dot{\tau}_{[t,T]}<T\}} \right) \right] \\
&= \mathbb{E}_t^{\mathbb{P}} \left[P(H,T)^2 \right] \left(1 - 2\omega \, \mathbb{E}_t^{\mathbb{R}_2^T} \left[\mathbf{1}_{\{\tau_{[t,T]}<T\}} \right] + \omega^2 \, \mathbb{E}_t^{\mathbb{R}_2^T} \left[\mathbf{1}_{\{\tau_{[t,T]}<T, \dot{\tau}_{[t,T]}<T\}} \right] \right).
\end{aligned}
$$

Expected Option Return Simulation

Figure IA2: Surface of Annualized Log Expected Returns on ATM Calls

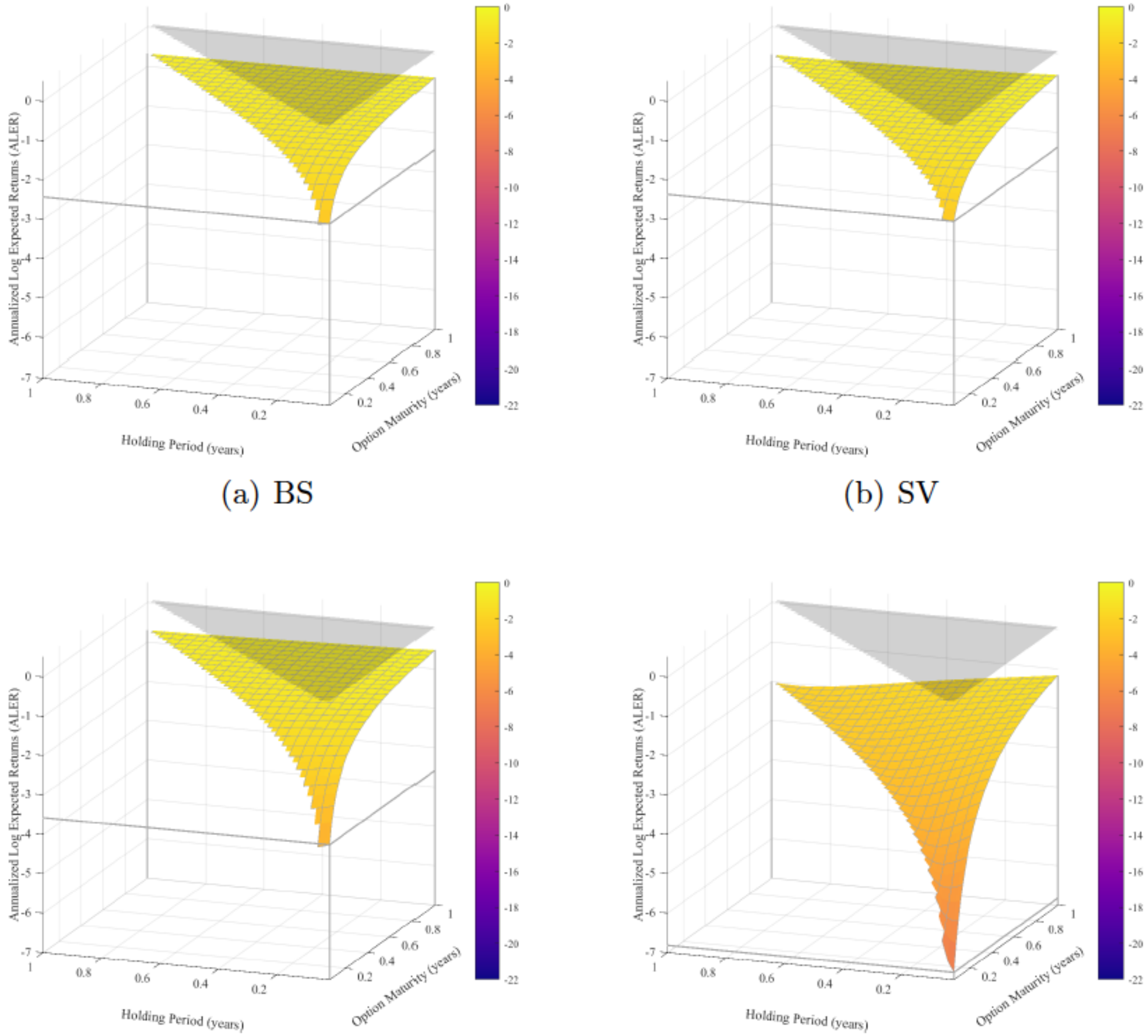


(a) BS (b) SV

(c) SVJ (d) SVJJ

Expected Option Return Simulation

Figure IA3: Surface of Annualized Log Expected Returns on ATM Puts



(a) BS (b) SV

(c) SVJ (d) SVJJ

Conclusion

- An important gap remains in the understanding of the expected returns and risks of contingent claims.

- We propose a theoretical framework to fill this gap:
  - Propose the equivalent expectation measures (EEMs) as generalizations of EMMs.
  - Propose R-transforms as generalizations of Q-transforms.
  - Solve the analytical solutions of expected prices, and higher moments under various types of contingent claim models using parallel processes.
  - Show how mean-variance efficient portfolios can be computed using a limited number of asset classes, which include both the options and the securities underlying these options.
  - Show how mean-variance efficient portfolios can be computed for subset of assets like Treasury Bonds and Corporate Bonds.
  - Future applications for parametric models using GMM and MCMC methods.

Thank you!