

A Theory of Equivalent Expectation Measures for Contingent Claim Returns

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ABSTRACT

This paper introduces a dynamic change of measure approach for computing the analytical solutions of expected future prices (and therefore, expected returns) of contingent claims over a finite horizon. The new approach constructs hybrid probability measures called the “equivalent expectation measures” (EEMs), which provide the physical expectation of the claim’s future price until before the horizon date, and serve as pricing measures on or after the horizon date. The EEM theory can be used for empirical investigations of both the cross-section and the term structure of returns of contingent claims, such as Treasury bonds, corporate bonds, and financial derivatives.

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The change of measure concept lies at the heart of asset pricing. Almost half-a-century ago, the no-arbitrage derivations of the call option price by Black and Scholes (1973) and Merton (1973b) led to the discovery of the standard change of measure approach, which assigns risk-neutral or equivalent martingale probabilities to all future events.¹ Extending this static approach which changes the pricing measure only at the current time, we develop a *dynamic* change of measure approach which changes the pricing measure at any given future horizon date H between the current time t and the claim's expiration date T . Using the new dynamic change of measure approach, this paper develops a theory for computing expected future prices, and therefore, expected returns of contingent claims over any finite horizon H , significantly broadening the range of applications of the change of measure approach in finance.

The theory developed in this paper can be used to answer the following kinds of questions. What is the expected return of a 10-year Treasury bond over the next two months under the $A_1(3)$ affine model of Dai and Singleton (2000)? What is the expected return of a 5-year, A-rated corporate bond over the next quarter under the stationary-leverage ratio model of Collin-Dufresne and Goldstein (2001)? What is the expected return of a 3-month call option on a stock over the next month under the SVJ model of Pan (2002) or under the CGMY model of Carr et al. (2002)? What is the expected return of a 5-year interest rate cap over next six months, under the QTSM3 interest rate model of Ahn et al. (2002)? While the analytical solutions of the current prices *exist* under all of the above-mentioned models, the analytical solutions of the expected future prices—which are necessary inputs for computing the expected returns—over an arbitrary finite horizon H , *do not exist* in the finance literature.

Estimating expected returns becomes difficult, if not impossible, for most buy-side firms and other buy-and-hold investors—who do not continuously rebalance and hedge their portfolios—without the analytical solutions of expected future prices of the financial securities in their portfolios. This problem is faced not just by sophisticated hedge fund managers who hold complex contingent claims,

¹See Cox and Ross (1976) and Harrison and Kreps (1979) for the development of the change of measure approach.

but also by numerous asset managers at mutual funds, banks, insurance companies, and pension funds, who hold the most basic financial securities, such as the U.S. Treasury bonds and corporate bonds. In an insightful study, Becker and Ivashina (2015) illustrate this problem in the context of insurance companies, whose asset managers reach for higher (promised) yields, and not necessarily higher expected returns. They find that the “yield-centered” holdings of insurance companies are related to the business cycle, being most pronounced during economic expansions. More generally, it is common to use metrics based upon yield, rating, sector/industry, optionality, and default probability, to make portfolio holding decisions by fixed income asset managers. Since expected returns are not modeled *explicitly*, these metrics provide at best only rough approximations, and at worst counter-intuitive guidance about the ex-ante expected returns as shown by Becker and Ivashina (2015). This is quite unlike the equity markets, where much effort is spent both by the academics and the portfolio managers for obtaining the conditional estimates of the expected returns on stocks. Part of this widespread problem in the fixed income markets and the financial derivatives markets is due to the lack of a simple and parsimonious framework that can guide the conditional estimation of finite-horizon expected returns. This paper fills this gap in the finance theory.

The intuition underlying our theory can be exemplified with the construction of a simple discrete-time example to compute the expected future price of a European call option. We begin by constructing a hybrid equivalent probability measure \mathbb{R} , which remains the physical measure \mathbb{P} until before time H , and becomes the risk-neutral measure \mathbb{Q} on or after time H , for a specific future horizon H between time t and time T . By construction the \mathbb{R} measure provides the physical expectation of the claim’s time- H future price until before time H , and serves as the pricing (or the equivalent martingale) measure on or after time H . Using this new measure, we show how to construct a binomial tree to obtain the expected future price the European call option C , which matures at time T with strike price K , written on an underlying asset price process S . Given $C_T = \max(S_T - K, 0) = (S_T - K)^+$ as the terminal payoff from the call option and using a constant interest rate r , the future price of the call option at time H can be computed under the \mathbb{Q} measure,

as follows:

$$C_H = \mathbb{E}_H^{\mathbb{Q}} [e^{-r(T-H)}(S_T - K)^+].$$

Taking the physical expectation of the future call price at the current time $t \leq H$, gives²

$$\mathbb{E}_t[C_H] = \mathbb{E}_t^{\mathbb{P}} [\mathbb{E}_H^{\mathbb{Q}} [e^{-r(T-H)}(S_T - K)^+]]. \quad (1)$$

Suppose we can force the construction of the new equivalent probability measure \mathbb{R} by satisfying the following two conditions:

- i) the \mathbb{P} transition probabilities at time t of all events until time H are equal to the corresponding \mathbb{R} transition probabilities of those events, and
- ii) the \mathbb{Q} transition probabilities at time H of all events until time T are equal to the corresponding \mathbb{R} transition probabilities of those events.

Using this construction, the law of iterated expectations immediately gives:

$$\mathbb{E}_t[C_H] = \mathbb{E}_t^{\mathbb{R}} [e^{-r(T-H)}(S_T - K)^+]. \quad (2)$$

Note that for each value of the horizon date H —from the current time t until the claim's maturity date T —equation (2) uses a different \mathbb{R} measure specific to that horizon. The stochastic process under the \mathbb{R} measure evolves under the physical measure \mathbb{P} until before time H , and under the risk-neutral measure \mathbb{Q} on or after time H .

Now, consider a discrete 2-year binomial tree illustrated in Figure 1, in which the current time $t = 0$, $H = 1$ years, $T = 2$ years, and the 1-year short rate is a constant.

To construct the discrete 2-year binomial tree for the stock price process under the \mathbb{R} measure, assume that the current price S_0 is \$100, the stock's annualized

²We use $\mathbb{E}[\cdot]$ and $\mathbb{E}^{\mathbb{P}}[\cdot]$ interchangeably to denote expectation under the physical measure \mathbb{P} . In addition, we denote by \mathbb{E}_t the conditional expectation with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, that is, $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$.

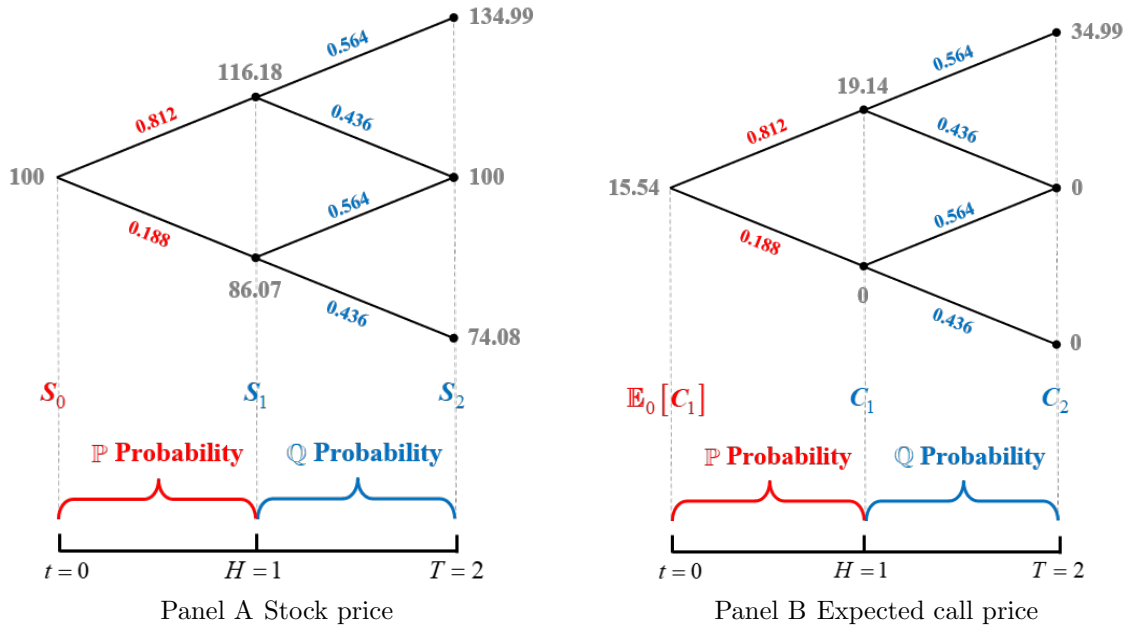


Figure 1: The binomial tree under the \mathbb{R} measure. This figure shows a 2-year binomial tree structure for the stock price and the expected call price under the \mathbb{R} measure. The \mathbb{R} measure follows physical \mathbb{P} distribution until before $H = 1$ years, and risk-neutral \mathbb{Q} distribution beginning at time $H = 1$ years. The following parameters are used: $S_t = 100$, $K = 100$, $\mu = 0.1$, $r = 0.03$, $\sigma = 0.15$, $\Delta t = 1$. The up and down node values of the future stock price evolution are computed using the binomial-tree method given by Cox et al. (1979). The up and down \mathbb{R} probabilities are the same as the corresponding up and down \mathbb{P} probabilities at time 0, equal to 0.812 and 0.188, respectively; and the up and down \mathbb{R} probabilities are the same as the corresponding up and down \mathbb{Q} probabilities at time 1, equal to 0.564 and 0.436, respectively.

expected return μ is 0.1, the annualized risk-free rate r is 0.03, the annualized stock return volatility σ is 0.15, and the discrete interval $\Delta t = 1$ year. The stock prices at different nodes are identical under all three measures, \mathbb{P} , \mathbb{Q} , and \mathbb{R} , and are given by the binomial model of Cox et al. (1979): the stock price either moves up by the multiplicative factor $u = e^{\sigma\sqrt{\Delta t}} = 1.1618$, or moves down by the multiplicative factor $d = e^{-\sigma\sqrt{\Delta t}} = 0.8607$ over each time step, as shown in Figure 1A.

The \mathbb{R} measure is constructed by satisfying conditions i) and ii) given above. To satisfy condition i), the next-period up and down \mathbb{R} probabilities are equal to

the corresponding up and down \mathbb{P} probabilities, respectively, at time $t = 0$, and are calculated as:

$$p^u = \frac{e^{\mu\Delta t} - d}{u - d} = \frac{e^{0.1 \times 1} - 0.8607}{1.1618 - 0.8607} = 0.812, \quad p^d = 1 - p^u = 0.188.$$

To satisfy condition ii), the next-period up and down \mathbb{R} probabilities are equal to the corresponding up and down \mathbb{Q} probabilities, respectively, at time $H = 1$, and are calculated as:

$$q^u = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.03 \times 1} - 0.8607}{1.1618 - 0.8607} = 0.564, \quad q^d = 1 - q^u = 0.436.$$

Now, consider the computation of the expected future price $\mathbb{E}_0[C_1]$ of a 2-year European call option written on this stock with a strike price $K = \$100$. The expected future call price can be computed using the \mathbb{R} measure as shown in Figure 1B.

The option prices C_2 at the terminal nodes of the tree are calculated by the payoff function $\max(S_2 - K, 0)$. The option prices C_1 at time $H = 1$ year are computed using risk-neutral discounting (see Cox et al., 1979). Specifically, the option value C_1 at the up node is

$$\begin{aligned} C_1^u &= e^{-r\Delta t} (q^u \times \$34.99 + q^d \times \$0) \\ &= e^{-0.03 \times 1} (0.564 \times \$34.99 + 0.436 \times \$0) = \$19.14. \end{aligned}$$

Similar calculations give $C_1^d = \$0$ at the down node. Then at the current time $t = 0$, the expected future call price $\mathbb{E}_0[C_1]$ is calculated by taking the expectation of call prices C_1^u and C_1^d , using the up and down \mathbb{P} probabilities, as follows:

$$\begin{aligned} \mathbb{E}_0[C_1] &= p^u \times C_1^u + p^d \times C_1^d \\ &= 0.812 \times \$19.14 + 0.188 \times \$0 = \$15.54. \end{aligned}$$

The above 2-year binomial tree can be generalized to multiple periods (of any length) with any specific horizon H between the current time and the option expiration date, by using the one-period up and down \mathbb{R} transition probabilities as the corresponding \mathbb{Q} transition probabilities for all periods between time H and

time T , and as the corresponding \mathbb{P} transition probabilities for all periods between time t and time H .

The above discrete-time example used the money market account as the numeraire and the risk-neutral measure \mathbb{Q} as the corresponding equivalent martingale measure for valuation on or after the horizon date H . The theory underpinning the above example can be generalized to continuous-time models that use alternative numeraires for the valuation of contingent claims. The mathematical intuition underlying this new theory can be understood as follows.

Given a numeraire G with the corresponding equivalent martingale measure (EMM) \mathbb{Q}^* and the Radon-Nikodým derivative process L^* , we show how the expected future prices of contingent claims can be computed using one of two new measures, \mathbb{R}^* and \mathbb{R}_1^* , with Radon-Nikodým derivative processes \mathcal{L}^* and \mathcal{L}_1^* , respectively. Each of the new measures is pieced together using two different Radon-Nikodým derivative processes, which break from each other at the horizon date H , so that the measure provides the physical expectation of the claim's time- H future price until before time H , and serves as the pricing (or the equivalent martingale) measure on or after time H . This simple and novel approach is shown to be very useful in providing formulas of the expected future prices of contingent claims, which are of the same general form as the corresponding valuation formulas of these claims, requiring that only the forms of the underlying price processes be similar under the physical measure and the pricing measure.

Unlike the traditional EMMs, discounted prices are martingales under the new measures, \mathbb{R}^* and \mathbb{R}_1^* , only from time H to time T , and not from time t to time H . Since these new measures provide the physical expectation of the claim's time- H future price until before time H , we denote these measures as *equivalent expectation measures (EEMs)*. The EMMs of Harrison and Kreps (1979) and Geman et al. (1995) are obtained as special cases of the corresponding EEMs, when $t = H$.

Of course, the dynamic change of measure is not the only method for computing the expected future price of a contingent claim. Two other methods which can be potentially used to compute the expected future price of a claim are the *stochastic discount factor* (e.g., see Hansen and Richard, 1987; Cochrane, 1996) method and the *numeraire change* method (e.g., see Long, 1990; Becherer, 2001; Karatzas and Kardaras, 2007). The future price of the call option at time H , can be expressed

under these two methods as follows:

$$C_H = \mathbb{E}_H^{\mathbb{P}} [M_T/M_H(S_T - K)^+] \quad (3)$$

$$= \mathbb{E}_H^{\mathbb{P}} [N_H/N_T(S_T - K)^+], \quad (3a)$$

where M is the pricing kernel, M_T/M_H is the future stochastic discount factor (SDF) at time H , and N is the numeraire portfolio using which makes all of the discounted prices \mathbb{P} martingales as shown by Long (1990). Taking iterated expectations under the \mathbb{P} measure, the time t expectation of the future call price is given as

$$\mathbb{E}_t[C_H] = \mathbb{E}_t^{\mathbb{P}} [M_T/M_H(S_T - K)^+] \quad (4)$$

$$= \mathbb{E}_t^{\mathbb{P}} [N_H/N_T(S_T - K)^+]. \quad (4a)$$

Equations (2), (4), and (4a) provide three distinct but related methods for computing the expected future price of the call option. Section I generalizes the dynamic change of method given by equation (2) to hold with stochastic interest rates and alternative numeraire assets. While the generalized dynamic change of method is mathematically equivalent to the *future* stochastic discount factor method given by equation (4), and the *dynamic* change of numeraire method given by equation (4a), the former method has a significant advantage over the latter two methods for the derivations of the analytical solutions of the expected future prices under virtually all contingent claim models that admit an analytical solution to their current price using an EMM.

Though this paper provides the first formal theory of the dynamic change of measure method, this is not the first paper to propose a change of measure at a future date. Rubinstein's (1984) formula for computing the expected return of a call option under the Black and Scholes (1973) model, and the valuation formulas for a forward-start option by Rubinstein (1991) and Kruse and Nögel (2005), also use a change of measure at a future date. These papers combine two pricing measures or EMMs, iteratively—one measure from time H until time T for doing valuation at time H , and another measure from the current time t until time H for computing the current expectation—which is a technique that

works under a limited set of models. In contrast, our paper creates a new class of hybrid measures called EEMs that ensure the derivations of analytical solutions to the expected future prices of contingent claims under virtually all models in the finance literature that admit an analytical solution to their current price using the existing EMMs.

Given the central role of expected returns in empirical asset pricing and in investment management, the additional flexibility provided by the analytical solutions of the expected future prices of contingent claims over different future horizons can significantly enhance our understanding of the investment opportunity set available in the Treasury bond market, corporate bond market, and virtually all of the financial derivatives in the equity, interest rate, foreign exchange, commodity, credit risk, and other areas of finance. A general econometric implication of the EEM theory is that the analytical solutions of the expected returns over different horizons can allow applications of various empirical estimation methods, such as linear regression, the Markov Chain Monte Carlo method, the generalized method of moments (see Hansen, 1982), and others for studying both the cross-section and the term structure of contingent claims returns in different markets.

For example, future research can evaluate the return prediction ability of different structural models for corporate bonds using model-implied³ analytical solutions of the expected returns, extending the empirical investigations of Gebhardt et al. (2005), Lin et al. (2011), Bai et al. (2019), Chung et al. (2019), and Bali et al. (2021). Similarly, future research can evaluate the return prediction ability of different dynamic term structure models (DTSMs) for Treasury bonds using the analytical solutions of the expected returns, extending the empirical investigations of Dai and Singleton (2000, 2002), Ahn et al. (2002), Duffee (2002), Bansal and Zhou (2002), Hong and Li (2005), Collin-Dufresne et al. (2008), and Joslin et al. (2011). To the best of our knowledge, Adrian et al. (2013) is the only existing study that directly incorporates the bond return information for model estimation using regressions of one-period excess returns. The availability of the analytical solutions of the expected bond returns over different horizons can also provide new

³For example, see Merton (1974), Longstaff et al. (2005), Leland and Toft (1996), and Collin-Dufresne and Goldstein (2001).

insights on the *term structure* of bond returns. While it may not be possible to study the term structure of corporate bond returns due to paucity of data and liquidity issues over short horizons, the DTSMs can be adapted for Treasury bond return prediction by forcing them to match the term structure of returns instead of the term structure of zero-coupon yields.

In the options area, the availability of the analytical solutions of the expected option returns for different horizons may not only advance our knowledge about the term structure of option returns, but also provide additional insight into the debate on whether index options are mispriced. For example, based on very high negative index put option expected returns over a *monthly* horizon, Coval and Shumway (2001) and Chambers et al. (2014) argue for mispricing, while Broadie et al. (2009) and Baele et al. (2019) argue for no mispricing. It is an open question whether using the analytical solutions of the expected option returns over different return horizons of 1 day, 2 day, 1 week, 2 weeks, up to one month, would add more insight on this ongoing empirical debate.

Last but not least, as an empirical application in the risk management area, the framework of this paper can also be extended for the computation of variance, covariance, and other higher order moments, and various tail risk measures used routinely for the risk and return analysis of fixed income portfolios and equity portfolios embedded with options.

This paper is organized as follows. Section I develops the theory of two general EEMs, \mathbb{R}^* and \mathbb{R}_1^* , using the dynamic change of measure method. These two measures lead to three numeraire-specific EEMs given as: i) the \mathbb{R} measure from the \mathbb{R}^* class using the money market account as the numeraire; ii) the \mathbb{R}_1^T measure from the \mathbb{R}_1^* class using the T -maturity pure discount bond as the numeraire; and iii) the \mathbb{R}_1^S measure from the \mathbb{R}_1^* class using any general asset S other than the money market account and the T -maturity pure discount bond as the numeraire. These three numeraire-specific EEMs allow the derivation of analytical solutions to the expected future prices of virtually all contingent claims in finance that are priced using the change of measure method.

Section II illustrates the use of \mathbb{R} measure with the examples of the Black and Scholes (1973) option pricing model and the affine term structure models of Dai and Singleton (2000) and Collin-Dufresne et al. (2008). This section also presents

new R -transforms based upon the \mathbb{R} measure, which can be used for obtaining the analytical solutions of the expected future prices of contingent claims that are priced using the transform-based models of Bakshi and Madan (2000), Duffie et al. (2000), and Chacko and Das (2002), and the CGMY Lévy model of Carr et al. (2002).

Section III illustrates the use of \mathbb{R}_1^T measure with the examples of the Merton (1973b) option pricing model and the structural debt pricing model of Collin-Dufresne and Goldstein (2001). This section also extends the theoretical framework of Breeden and Litzenberger (1978) to obtain the \mathbb{R}_1^T density and the physical density of the underlying asset using non-parametric and semi-parametric empirical models of expected option returns.

Internet Appendix Sections II, III, IV, V, and our ongoing/future research (see Nawalkha and Zhuo, 2022) derive the expected future prices of different contingent claims under various classes of models in the equity, interest rate, and credit risk areas using the \mathbb{R} and the \mathbb{R}_1^T measures.⁴ Internet Appendix Section VI illustrates the use of \mathbb{R}_1^S measure with an extension of the Margrabe (1978) exchange option pricing model with stochastic interest rates. We show that while the valuation of exchange option does not depend upon interest rates under the Margrabe (1978) model, the expected future price of the exchange option does depend upon the parameters of the interest rate process.

The final section considers some theoretical extensions of the EEM theory and provides concluding remarks. Some of the proofs of the theorems and the propositions are provided in the Appendix, and some of the models are presented

⁴In these Internet Appendix sections, we illustrate additional applications of the EEMs \mathbb{R} and \mathbb{R}_1^T for obtaining the analytical solutions of the expected future prices of contingent claims under the specific models of Vasicek (1977), Cox et al. (1985), the $A_{1r}(3)$ model of Balduzzi et al. (1996) and Dai and Singleton (2000), the general quadratic model of Ahn et al. (2002) with the specific example of the QTSM3 model, the Merton's (1973b) model with the Vasicek (1977) short rate process embedded in the Merton's (1973a) ICAPM framework, and the forward-start option model using the affine jump-diffusion processes of Duffie et al. (2000). Using the R -transforms presented in Section II.C, Nawalkha and Zhuo (2022) derive the analytical solutions of the expected future prices of European options under the stochastic-volatility-double-jump model of Duffie et al. (2000), the CGMY Lévy model of Carr et al. (2002), and interest rate derivatives under the affine jump-diffusion-based models of Chacko and Das (2002).

in the Internet Appendix.

I. The Equivalent Expectation Measure Theory

This section builds the theory of two general EEMs \mathbb{R}^* and \mathbb{R}_1^* . Identifying these general EEMs with specific numeraires, such as the money market account, the T -maturity pure discount bond, and a general traded asset, leads to different numeraire-specific EEMs, which can be used to obtain expected future prices of contingent claims under virtually all models that admit an analytical solution to the claim's current price.

A. The \mathbb{R}^* Measure

Let the price of a contingent claim be denoted by F , with a maturity/expiration date T on which the claim pays out a single contingent cash flow. The cash flow can depend upon the entire history of the underlying state variables in arbitrary ways (as in the papers of Harrison and Kreps, 1979; Bakshi and Madan, 2000; Duffie et al., 2000; Chacko and Das, 2002), allowing a variety of regular and exotic contingent claims, such as European options, Asian options, barrier options, and other claims, which make a single payment at a fixed terminal date T .

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration \mathcal{F}_s , $0 \leq s \leq T$, satisfying the usual conditions,⁵ where \mathbb{P} is the physical probability measure. We assume that the contingent claim is being valued by an EMM \mathbb{Q}^* defined on (Ω, \mathcal{F}) , which uses a traded asset G as the numeraire for valuing the claim.⁶ According to the martingale valuation results (see, e.g., Geman et al., 1995; Björk, 2009, Theorem 26.2), the process F_s/G_s is a martingale under \mathbb{Q}^* for all $0 \leq s \leq T$, and the current time t price of the claim is given as

$$F_t = G_t \mathbb{E}_t^{\mathbb{Q}^*} \left[\frac{F_T}{G_T} \right]. \quad (5)$$

⁵ \mathcal{F}_s is right-continuous and \mathcal{F}_0 contains all the P -negligible events in \mathcal{F} .

⁶The numeraire G could be the money market account, zero-coupon bond, or any other asset, for instance, stock, commodity, etc.

Due to the equivalence of \mathbb{Q}^* and \mathbb{P} , there exists an almost surely positive random variable L_T^* such that

$$\mathbb{Q}^*(A) = \int_A L_T^*(\omega) d\mathbb{P}(\omega) \quad \text{for all } A \in \mathcal{F}_T,$$

where L_T^* is the Radon-Nikodým derivative of \mathbb{Q}^* with respect to \mathbb{P} , and we can write it as

$$L_T^* = \left. \frac{d\mathbb{Q}^*}{d\mathbb{P}} \right|_{\mathcal{F}_T},$$

with $\mathbb{E}[L_T^*] = 1$. The Radon-Nikodým derivative process is defined as

$$L_s^* = \mathbb{E}[L_T^* | \mathcal{F}_s], \quad \text{or } L_s^* = \left. \frac{d\mathbb{Q}^*}{d\mathbb{P}} \right|_{\mathcal{F}_s}, \quad \text{for } 0 \leq s \leq T.$$

The *expected future price* of the claim, $\mathbb{E}_t[F_H]$, can be solved using equation (5) at a future time H , as follows:

$$\mathbb{E}_t[F_H] = \mathbb{E}_t^{\mathbb{P}} \left[G_H \mathbb{E}_H^{\mathbb{Q}^*} \left[\frac{F_T}{G_T} \right] \right]. \quad (6)$$

Similar to equation (1) in the introduction, the main reason for the intractability of equation (6) is that the law of iterated expectations cannot be used since the inner and the outer expectations are not under the same probability measure. In order to make the inner and the outer expectations under the same probability measure, we define an equivalent probability measure \mathbb{R}^* in the following Lemma 1, under which the following two conditions are met for all $0 \leq t \leq H \leq T$:

- C1.** The \mathbb{P} transition probabilities at time t of all events until time H are equal to the corresponding \mathbb{R}^* transition probabilities of those events.
- C2.** The \mathbb{Q}^* transition probabilities at time H of all events until time T are equal to the corresponding \mathbb{R}^* transition probabilities of those events.

If both the above conditions are satisfied, then equation (6) can be simplified

using iterated expectations under the \mathbb{R}^* measure as follows:

$$\begin{aligned}\mathbb{E}_t[F_H] &= \mathbb{E}_t^{\mathbb{R}^*} \left[G_H \mathbb{E}_H^{\mathbb{R}^*} \left[\frac{F_T}{G_T} \right] \right] \\ &= \mathbb{E}_t^{\mathbb{R}^*} \left[\mathbb{E}_H^{\mathbb{R}^*} \left[G_H \frac{F_T}{G_T} \right] \right] \\ &= \mathbb{E}_t^{\mathbb{R}^*} \left[G_H \frac{F_T}{G_T} \right].\end{aligned}\tag{7}$$

To this end, we construct the \mathbb{R}^* measure using the following lemma:

LEMMA 1: For a fixed H with $0 \leq H \leq T$, define a process $\mathcal{L}_s^*(H)$ as

$$\mathcal{L}_s^*(H) = \begin{cases} \frac{L_s^*}{L_H^*}, & \text{if } H \leq s \leq T, \\ 1, & \text{if } 0 \leq s < H. \end{cases}\tag{8}$$

Let

$$\mathbb{R}^*(A) = \int_A \mathcal{L}_T^*(H; \omega) d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F}_T,$$

then \mathbb{R}^* is a probability measure equivalent to \mathbb{P} , and $\mathcal{L}_s^*(H)$ is the Radon-Nikodým derivative process of \mathbb{R}^* with respect to \mathbb{P} .

Proof. See Appendix A (Section A). □

We now state the main theorem of this paper.

THEOREM 1: For all $0 \leq t \leq H \leq T$, the time t expectation of the claim's future price F_H can be computed by using the equivalent expectation measure \mathbb{R}^* defined by Lemma 1, as shown in equation (7).

Proof. Since equation (7) follows from equation (6) if conditions C1 and C2 are satisfied, the proof requires us to show that the \mathbb{R}^* measure defined in Lemma 1 satisfies both these conditions. For the first condition C1, note that the Radon-Nikodým derivative process of \mathbb{R}^* w.r.t. \mathbb{P} equals 1 before time H , which implies that $\mathbb{R}^* = \mathbb{P}$, before H . For the second condition C2, note that for any event, say $A_T \in \mathcal{F}_T$, its conditional \mathbb{Q}^* probability at time H is the same as the corresponding

\mathbb{R}^* probability as follows:

$$\mathbb{E}_H^{\mathbb{Q}^*} [\mathbb{1}_{A_T}] = \mathbb{E}_H^{\mathbb{P}} \left[\frac{L_T^*}{L_H^*} \mathbb{1}_{A_T} \right] = \mathbb{E}_H^{\mathbb{P}} \left[\frac{L_T^*/L_H^*}{L_H^*/L_H^*} \mathbb{1}_{A_T} \right] = \mathbb{E}_H^{\mathbb{P}} \left[\frac{\mathcal{L}_T^*(H)}{\mathcal{L}_H^*(H)} \mathbb{1}_{A_T} \right] = \mathbb{E}_H^{\mathbb{R}^*} [\mathbb{1}_{A_T}].$$

As an alternative proof we can derive equation (7) directly from equation (6) by using Lemma 1 (and without explicitly invoking conditions C1 and C2, which are implied by Lemma 1) as follows:

$$\begin{aligned} \mathbb{E}_t [F_H] &= \mathbb{E}_t^{\mathbb{P}} \left[\mathbb{E}_H^{\mathbb{Q}^*} \left[G_H \frac{F_T}{G_T} \right] \right] = \mathbb{E}_t^{\mathbb{P}} \left[\mathbb{E}_H^{\mathbb{P}} \left[\frac{L_T^*}{L_H^*} \cdot G_H \frac{F_T}{G_T} \right] \right] \\ &= \mathbb{E}_t^{\mathbb{P}} \left[\frac{L_T^*}{L_H^*} \cdot G_H \frac{F_T}{G_T} \right] = \mathbb{E}_t^{\mathbb{P}} \left[\frac{L_T^*/L_H^*}{1} \cdot G_H \frac{F_T}{G_T} \right] \\ &= \mathbb{E}_t^{\mathbb{P}} \left[\frac{\mathcal{L}_T^*(H)}{\mathcal{L}_t^*(H)} \cdot G_H \frac{F_T}{G_T} \right] = \mathbb{E}_t^{\mathbb{R}^*} \left[G_H \frac{F_T}{G_T} \right]. \end{aligned} \tag{9}$$

□

We now give some general properties of the \mathbb{R}^* measure that provide insights into its relationship with the physical measure \mathbb{P} and the equivalent martingale measure \mathbb{Q}^* . We begin with Proposition 1, which gives the relationship of the \mathbb{R}^* measure with the \mathbb{P} and \mathbb{Q}^* measures.

PROPOSITION 1: *For all $0 \leq t \leq H \leq s \leq T$, the \mathbb{R}^* measure has the following properties:*

- (i) $\mathbb{R}^*(A | \mathcal{F}_t) = \mathbb{P}(A | \mathcal{F}_t)$ for all $A \in \mathcal{F}_H$.
- (ii) $\mathbb{R}^*(A | \mathcal{F}_s) = \mathbb{Q}^*(A | \mathcal{F}_s)$ for all $A \in \mathcal{F}_T$.
- (iii) When $H = T$, $\mathbb{R}^*(A | \mathcal{F}_t) = \mathbb{P}(A | \mathcal{F}_t)$ for all $A \in \mathcal{F}_T$.
- (iv) When $H = t$, $\mathbb{R}^*(A | \mathcal{F}_t) = \mathbb{Q}^*(A | \mathcal{F}_t)$ for all $A \in \mathcal{F}_T$.
- (v) $\mathbb{E}_t^{\mathbb{R}^*} [Z_T] = \mathbb{E}_t^{\mathbb{P}} \left[\mathbb{E}_H^{\mathbb{Q}^*} [Z_T] \right]$ for any random variable Z_T at time T .

Proof. See Appendix A (Section B). □

According to Proposition 1(i), the conditional \mathbb{R}^* probabilities at any time t until time H , of the events at time H , are the same as the corresponding \mathbb{P}

probabilities of those events, satisfying condition C1 given earlier. According to Proposition 1(ii), the conditional \mathbb{R}^* probabilities at any time s on or after time H , of the events at time T , are the same as the corresponding conditional \mathbb{Q}^* probabilities of those events, satisfying condition C2 given earlier. Propositions 1(iii) and 1(iv) show that the \mathbb{R}^* measure nests both the physical measure \mathbb{P} and the equivalent martingale measure \mathbb{Q}^* . When $H = t$, the \mathbb{R}^* measure becomes the \mathbb{Q}^* measure, and the expected value computing problem becomes the traditional claim pricing problem. When $H = T$, the \mathbb{R}^* measure becomes the \mathbb{P} measure. For all other values of H that are strictly between the current time t and the maturity/expiration date T , the \mathbb{R}^* measure remains a hybrid measure. Proposition 1(v) shows that the expectation of any random variable Z_T under the \mathbb{R}^* measure is equivalent to the iterated expectation of that variable computed using both \mathbb{P} and \mathbb{Q}^* measures.

PROPOSITION 2: *The expectation of the price ratio process F_s/G_s has the following properties:*

- (i) For all $0 \leq H \leq s_1 \leq s_2 \leq T$, $\mathbb{E}_{s_1}^{\mathbb{R}^*} [F_{s_2}/G_{s_2}] = \mathbb{E}_{s_1}^{\mathbb{Q}^*} [F_{s_2}/G_{s_2}] = F_{s_1}/G_{s_1}$.
- (ii) For all $0 \leq s_1 \leq s_2 \leq H \leq T$, $\mathbb{E}_{s_1}^{\mathbb{R}^*} [F_{s_2}/G_{s_2}] = \mathbb{E}_{s_1}^{\mathbb{P}} [F_{s_2}/G_{s_2}] \neq \mathbb{E}_{s_1}^{\mathbb{Q}^*} [F_{s_2}/G_{s_2}] = F_{s_1}/G_{s_1}$.

Proof. See Appendix A (Section C). □

According to Proposition 2(i), the price ratio process F_s/G_s is a martingale under \mathbb{R}^* from time H to time T . However, according to Proposition 2(ii), unless $\mathbb{P}(A) = \mathbb{Q}^*(A)$ for all $A \in \mathcal{F}_H$, the price ratio process F_s/G_s is not a martingale under \mathbb{R}^* from time 0 to time H . This shows the essential difference between the equivalent *martingale* measure \mathbb{Q}^* under which the price ratio F_s/G_s is a martingale from time 0 until time T , and the corresponding equivalent *expectation* measure \mathbb{R}^* under which F_s/G_s is a martingale only from time H until time T .

Generalizing the results in equations (4) and (4a), the expected future price of the claim in equation (7) can also be expressed as follows:

$$\mathbb{E}_t[F_H] = \mathbb{E}_t^{\mathbb{P}} \left[\frac{M_T}{M_H} F_T \right] \quad (10)$$

$$= \mathbb{E}_t^{\mathbb{P}} \left[\frac{N_H}{N_T} F_T \right], \quad (10a)$$

where the future SDF M_T/M_H equals N_H/N_T , the inverse of the future gross return on the numeraire portfolio from time H to time T (see Long, 1990), for all $0 \leq t \leq H \leq T$. The equivalence relationship for the expected future price of a contingent claim given in equations (7), (10), and (10a) connects the new dynamic change of measure method to two asset pricing literatures in finance on the SDF-based models and the numeraire change-based models. If the numeraire G is selected as the numeraire portfolio N of Long (1990), then both \mathbb{Q}^* and \mathbb{R}^* become the physical measure \mathbb{P} , and equation (10a) obtains as a special case of equation (7). If the numeraire G is selected as the money market account, then \mathbb{Q}^* becomes the risk-neutral measure \mathbb{Q} , and \mathbb{R}^* becomes the \mathbb{R} measure illustrated in the introduction.

To evaluate the usefulness of the dynamic change of measure method relative to the future SDF method given by equation (10) and the dynamic change of numeraire method given by equation (10a), first consider those contingent claim models under which G_H/G_T is distributed independently of F_T , such that the expectation in equation (7) can be given as a product of two expectations that are easier to derive and compute. For example, consider all models which use the money market account as the numeraire, and assume a constant or deterministic short rate. The expectation in equation (7) under these models depends upon only the distribution of F_T under the \mathbb{R} measure, since G_H/G_T is a constant.⁷ In contrast, the expectations using equations (10) and (10a) depend upon the interaction of F_T with either M_T/M_H or with N_H/N_T under the \mathbb{P} measure. Since F_T is almost never independent of either M_T/M_H or N_H/N_T , the analytical solutions are easier to derive using the dynamic change of measure method given by equation (7). This advantage of the dynamic change of measure method is especially relevant when the underlying state variable processes follow complex stochastic processes, such as under the constant-elasticity-of-variance models of Cox and Ross (1976), the stochastic volatility models of Hull and White (1987) and Heston (1993), and the stochastic-volatility-jump models of Duffie et al. (2000) and Pan (2002). In-

⁷By definition, $\mathbb{R}^* = \mathbb{R}$, when the numeraire is the money market account.

tuitively, the same reasoning explains the widespread adoption of the risk-neutral measure \mathbb{Q} for contingent claims valuation, even though the SDF-based approach and the numeraire change approach can also be used.

Next, consider those contingent claim models under which G_H/G_T is stochastic and F_T is constant in equation (7). These models comprise the dynamic term structure models for pricing default-free bonds with the numeraire G as the money market account. Since a large empirical literature exists on these term structure models which estimates the state variables processes under both \mathbb{P} and \mathbb{Q} measures, the state variable processes under the \mathbb{R} measure are readily available. Moreover, the techniques to derive the analytical solutions of the expected future price using the dynamic change of measure method given by equation (7) follow as straightforward extensions of the existing techniques given by Dai and Singleton (2000), Ahn et al. (2002), and Collin-Dufresne et al. (2008), as shown in Section II.B.

Finally, consider the contingent claim models under which G_H/G_T and F_T are not independently distributed. For these models we propose the \mathbb{R}_1^* measure in the next section, which breaks the expectation of the future price of the claim in equation (7) into a product of two expectations. Doing this simplifies the derivation of the analytical solution of the expected future price of the claim considerably under models that use numeraires other than the money market account for pricing the claim.

B. The \mathbb{R}_1^* Measure

Consider separating the single expectation of the expression $G_H F_T / G_T$ in equation (7) into a product of *two* expectations, and define another equivalent expectation measure \mathbb{R}_1^* , as follows:

$$\begin{aligned} \mathbb{E}_t [F_H] &= \mathbb{E}_t^{\mathbb{R}^*} [G_H] \mathbb{E}_t^{\mathbb{R}_1^*} \left[\frac{F_T}{G_T} \right] \\ &= \mathbb{E}_t^{\mathbb{P}} [G_H] \mathbb{E}_t^{\mathbb{R}_1^*} \left[\frac{F_T}{G_T} \right], \end{aligned} \quad (11)$$

where for any \mathcal{F}_T -measurable variable Z_T , \mathbb{R}_1^* is defined with respect to \mathbb{R}^* as follows:

$$\mathbb{E}_t^{\mathbb{R}_1^*}[Z_T] = \mathbb{E}_t^{\mathbb{R}^*} \left[\frac{G_H}{\mathbb{E}_t^{\mathbb{R}^*}[G_H]} \cdot Z_T \right]. \quad (12)$$

The existence of the \mathbb{R}^* measure guarantees the existence of the \mathbb{R}_1^* measure as defined in equation (12). We construct the \mathbb{R}_1^* measure with respect to \mathbb{P} , using the following lemma.

LEMMA 2: For a fixed H with $0 \leq H \leq T$, define a process $\mathcal{L}_{1s}^*(H)$ as

$$\mathcal{L}_{1s}^*(H) = \begin{cases} \frac{G_H}{\mathbb{E}_0^{\mathbb{P}}[G_H]} \cdot \frac{L_s^*}{L_H^*}, & \text{if } H \leq s \leq T, \\ \frac{\mathbb{E}_s^{\mathbb{P}}[G_H]}{\mathbb{E}_0^{\mathbb{P}}[G_H]}, & \text{if } 0 \leq s < H. \end{cases}$$

Let

$$\mathbb{R}_1^*(A) \triangleq \int_A \mathcal{L}_{1T}^*(H; \omega) d\mathbb{P}(\omega) \quad \text{for all } A \in \mathcal{F}_T,$$

then \mathbb{R}_1^* is a probability measure equivalent to \mathbb{P} , and $\mathcal{L}_{1s}^*(H)$ is the Radon-Nikodým derivative process of \mathbb{R}_1^* with respect to \mathbb{P} .

Proof. See Appendix A (Section D). □

The next theorem proves that the definition of the \mathbb{R}_1^* measure given in Lemma 2 is consistent with the expectation result given by equation (11).

THEOREM 2: For all $0 \leq t \leq H \leq T$, the time t expectation of the claim's future price F_H can be computed by using the equivalent expectation measure \mathbb{R}_1^* defined by Lemma 2, as shown in equation (11).

Proof. The proof of equation (11) follows from the third equality in equation (9)

and applying Lemma 2, as follows:

$$\begin{aligned}
\mathbb{E}_t[F_H] &= \mathbb{E}_t^{\mathbb{P}} \left[\frac{L_T^*}{L_H^*} \cdot G_H \frac{F_T}{G_T} \right] = \mathbb{E}_t^{\mathbb{P}} [G_H] \mathbb{E}_t^{\mathbb{P}} \left[\frac{G_H}{\mathbb{E}_t^{\mathbb{P}} [G_H]} \cdot \frac{L_T^*}{L_H^*} \cdot \frac{F_T}{G_T} \right] \\
&= \mathbb{E}_t^{\mathbb{P}} [G_H] \mathbb{E}_t^{\mathbb{P}} \left[\frac{G_H / \mathbb{E}_0 [G_H] \cdot L_T^* / L_H^* F_T}{\mathbb{E}_t^{\mathbb{P}} [G_H] / \mathbb{E}_0 [G_H] G_T} \right] \\
&= \mathbb{E}_t^{\mathbb{P}} [G_H] \mathbb{E}_t^{\mathbb{P}} \left[\frac{\mathcal{L}_{1T}^*(H)}{\mathcal{L}_{1t}^*(H)} \cdot \frac{F_T}{G_T} \right] \\
&= \mathbb{E}_t^{\mathbb{P}} [G_H] \mathbb{E}_t^{\mathbb{R}_1^*} \left[\frac{F_T}{G_T} \right].
\end{aligned}$$

In addition, it is easy to verify that the mathematical construction of the \mathbb{R}_1^* measure in Lemma 2 is consistent with its proposed definition in equation (12) as follows.

For any \mathcal{F}_T -measurable variable Z_T , we have

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{R}_1^*} [Z_T] &= \mathbb{E}_t^{\mathbb{P}} \left[\frac{\mathcal{L}_{1T}^*(H)}{\mathcal{L}_{1t}^*(H)} Z_T \right] = \mathbb{E}_t^{\mathbb{P}} \left[\frac{G_H / \mathbb{E}_0 [G_H] \cdot L_T^* / L_H^* Z_T}{\mathbb{E}_t^{\mathbb{P}} [G_H] / \mathbb{E}_0 [G_H]} \right] \\
&= \mathbb{E}_t^{\mathbb{P}} \left[\frac{L_T^*}{L_H^*} \cdot \frac{G_H}{\mathbb{E}_t^{\mathbb{P}} [G_H]} Z_T \right] = \mathbb{E}_t^{\mathbb{P}} \left[\frac{\mathcal{L}_T^*(H)}{\mathcal{L}_t^*(H)} \cdot \frac{G_H}{\mathbb{E}_t^{\mathbb{P}} [G_H]} Z_T \right] \quad (13) \\
&= \mathbb{E}_t^{\mathbb{R}^*} \left[\frac{G_H}{\mathbb{E}_t^{\mathbb{P}} [G_H]} Z_T \right] = \mathbb{E}_t^{\mathbb{R}^*} \left[\frac{G_H}{\mathbb{E}_t^{\mathbb{R}^*} [G_H]} Z_T \right],
\end{aligned}$$

where the last equality follows since $\mathbb{E}_t^{\mathbb{R}^*} [G_H] = \mathbb{E}_t^{\mathbb{P}} [\mathcal{L}_H^*(H) / \mathcal{L}_t^*(H) \cdot G_H] = \mathbb{E}_t^{\mathbb{P}} [G_H]$. \square

Internet Appendix Section I.A gives some general properties of the \mathbb{R}_1^* measure that provide additional insights into its relationship with the physical measure \mathbb{P} and the equivalent martingale measure \mathbb{Q}^* . According to these properties, the conditional \mathbb{R}_1^* probabilities at any time s on or after time H , of the events at time T , are the same as the corresponding conditional \mathbb{R}^* probabilities, as well as the conditional \mathbb{Q}^* probabilities of those events. However, the conditional \mathbb{R}_1^* probabilities at any time t until time H , of any events at time H , are not the same as the corresponding \mathbb{P} probabilities of those events, due to an adjustment term equal to $G_H / \mathbb{E}_t^{\mathbb{P}} [G_H]$, related to the numeraire G .

The separation of the expectation of the expression $G_H F_T / G_T$ in equation (7)

into a product of two expectations in equation (11) is useful for obtaining the analytical solution of the expected future price of a claim when G_H/G_T and F_T are not distributed independently of each other.⁸ We demonstrate applications of equation (11) using the Merton's (1973b) stochastic interest rate-based option pricing model and Collin-Dufresne and Goldstein's (2001) corporate bond pricing model in Section III, and Margrabe's (1978) asset-exchange option model in Internet Appendix Section VI.

Under many contingent claim models, such as Merton (1973b), Margrabe (1978), and Grabbe (1983), the solution of $\mathbb{E}_t^{\mathbb{R}_1^*} [F_T/G_T]$ in equation (11) requires the distribution of the normalized asset price S_T/G_T , under the \mathbb{R}_1^* measure, where S_T is the time- T price of the asset underlying the claim. The expectation of S_T/G_T under the \mathbb{R}_1^* measure can be solved by substituting $F_T = S_T$ in equation (11), as follows:

$$\mathbb{E}_t^{\mathbb{R}_1^*} \left[\frac{S_T}{G_T} \right] = \frac{\mathbb{E}_t^{\mathbb{P}} [S_H]}{\mathbb{E}_t^{\mathbb{P}} [G_H]}. \quad (14)$$

The derivations of the analytical solutions to the expected future prices of contingent claims are simplified considerably using the result in equation (14), since the

⁸Of course, equation (11) is not the only way to separate the expectation of the expression $G_H F_T/G_T$ in equation (7) into a product of two or more expectations. Another way may be to separate this expectation into a product of the expectations of G_H/G_T and F_T using another EEM measure obtained as yet another transformation of the \mathbb{R}^* measure, similar to the transformation shown in equation (12). This can be done as follows: Let $\mathbb{E}_t [F_H] = \mathbb{E}_t^{\mathbb{R}^*} \left[\frac{G_H}{G_T} \right] \mathbb{E}_t^{\mathbb{R}_2^*} [F_T]$, where for any \mathcal{F}_T -measurable variable Z_T , \mathbb{R}_2^* is defined with respect to \mathbb{R}^* as follows:

$$\mathbb{E}_t^{\mathbb{R}_2^*} [Z_T] = \mathbb{E}_t^{\mathbb{R}^*} \left[\frac{G_H/G_T}{\mathbb{E}_t^{\mathbb{R}^*} [G_H/G_T]} \cdot Z_T \right].$$

We show in Internet Appendix Section I.B that if the numeraires are restricted to be either the money market account or the pure discount bond, then the \mathbb{R}_1^* measure subsumes the \mathbb{R}_2^* measure. A final way may be to separate this expectation into a product of three expectations of G_H , F_T , and $1/G_T$ using even more EEMs. However, we find that the expectations given under the two EEMs \mathbb{R}^* and \mathbb{R}_1^* by equations (7) and (11), respectively, are *sufficient* for obtaining the analytical solutions of the expected future prices of contingent claims under all models in finance that admit analytical solutions to the current prices of these claims using the change of measure method.

physical expectations of S_H and G_H are typically available in closed-form.

The \mathbb{R}^* measure given by Lemma 1 defines three numeraire-specific EEMs: \mathbb{R} , \mathbb{R}^T , and \mathbb{R}^S , corresponding to the three numeraires: the money market account, the T -maturity pure discount bond, and any tradable asset S other than the money market account and the T -maturity pure discount bond, respectively. Similarly, the \mathbb{R}_1^* measure given by Lemma 2 defines three numeraire-specific EEMs: \mathbb{R}_1 , \mathbb{R}_1^T , and \mathbb{R}_1^S , corresponding to the above three numeraires, respectively. Out of these six EEMs, we find that only three EEMs are useful for obtaining the analytical solutions of the expected future prices of contingent claims, given as follows:

- i) the \mathbb{R} measure from the \mathbb{R}^* class, obtained as a generalization of the risk-neutral measure \mathbb{Q} given by Black and Scholes (1973), Cox and Ross (1976), and Harrison and Kreps (1979);
- ii) the \mathbb{R}_1^T measure from the \mathbb{R}_1^* class, obtained as a generalization of the forward measure \mathbb{Q}^T given by Merton (1973b),⁹ Geman (1989), Jamshidian (1989), and Geman et al. (1995);
- iii) the \mathbb{R}_1^S measure from the \mathbb{R}_1^* class, obtained as a generalization of the equivalent martingale measure \mathbb{Q}^S , where the numeraire that defines \mathbb{Q}^S is any tradable asset S other than the money market account and the T -maturity pure discount bond (e.g., see Margrabe (1978), Geman et al. (1995), Rady (1997), and the examples in Vecer (2011), among others).

We demonstrate various applications of the \mathbb{R} measure and the \mathbb{R}_1^T measure in Sections II and III, respectively. Due to space constraints, we demonstrate the application of the \mathbb{R}_1^S measure in Internet Appendix Section VI using the example of the Margrabe (1978) exchange option pricing model.

II. The \mathbb{R} Measure

Assuming similar form for the physical and risk-neutral processes, the \mathbb{R} measure can be used for the derivation of the analytical solutions of expected future

⁹The call price expectation under the \mathbb{Q}^T measure is consistent with the Merton's (1973b) PDE for the call price using the Feynman-Kac theorem.

prices of contingent claims under all models that admit an analytical solution to the claim's price using the risk-neutral measure \mathbb{Q} . The following corollary to Theorem 1 can be used for all applications of the \mathbb{R} measure.

COROLLARY 1: *When the numeraire asset is the money market account, i.e., $G = B = e^{\int_0^T r_u du}$, then $\mathbb{R}^* = \mathbb{R}$, and the expected future price of the contingent claim G , can be given using equation (7) as*

$$\mathbb{E}_t [F_H] = \mathbb{E}_t^{\mathbb{R}} \left[e^{-\int_H^T r_u du} F_T \right]. \quad (15)$$

Proof. Equation (15) is a direct application of equation (7). □

The \mathbb{R} measure can be used to derive the expected future prices of contingent claims under the following models:¹⁰ the equity option pricing models of Black and Scholes (1973), Cox and Ross (1976), Merton (1976), Hull and White (1987), and Rubinstein (1991); the corporate debt pricing models of Merton (1974), Black and Cox (1976), and Leland and Toft (1996); the term structure models of Dai and Singleton (2000, 2002), Ahn et al. (2002), Leippold and Wu (2003), and Collin-Dufresne et al. (2008) for pricing default-free bonds; the credit default swap pricing model of Longstaff et al. (2005); the VIX futures and the variance swaps models of Dew-Becker et al. (2017), Eraker and Wu (2017), Johnson (2017) and Cheng (2019)¹¹; the currency option model of Garman and Kohlhagen (1983); and the various Fourier transform-based contingent claim models of Heston (1993), Duffie et al. (2000), Carr et al. (2002), and others, summarized in Section II.C, among others.

We present the \mathbb{R} measure using the examples of the Black and Scholes (1973) option pricing model, the affine term structure models of Dai and Singleton (2000) and Collin-Dufresne et al. (2008), and the Fourier transform-based model of Duffie

¹⁰We apply the \mathbb{R} measure to some of the models in the main part of the paper, and provide this measure under a multidimensional Brownian motion in Internet Appendix Section I.C, which can be applied to other models not explicitly considered in the main part of the paper. Some of these applications are given in the Appendix and the Internet Appendix. In our ongoing/future research (see Nawalkha and Zhuo, 2022), we also derive the \mathbb{R} measure for Lévy processes.

¹¹Eraker and Wu (2017) and Cheng (2019) consider only VIX futures and not variance swaps. Dew-Becker et al. (2017) and Johnson (2017) consider both VIX futures and variance swaps.

et al. (2000) in this section.

A. The Black-Scholes Model

The geometric Brownian motion for the asset price process under the Black and Scholes (1973) model is given as follows:

$$\frac{dS_s}{S_s} = \mu ds + \sigma dW_s^{\mathbb{P}}, \quad (16)$$

where μ is the drift, σ is the volatility, and $W^{\mathbb{P}}$ is the Brownian motion under the physical measure \mathbb{P} . The Girsanov theorem allows the transformation of Brownian motions under two equivalent probability measures. Defining $W_s^{\mathbb{Q}} = W_s^{\mathbb{P}} + \int_0^s \gamma du$, where $\gamma = (\mu - r)/\sigma$ is the market price of risk (MPR), the asset price process under the risk neutral measure \mathbb{Q} is

$$\frac{dS_s}{S_s} = r ds + \sigma dW_s^{\mathbb{Q}}. \quad (17)$$

Using Lemma 1, the Brownian motion under the \mathbb{R} measure is derived in Appendix B as follows:

$$W_s^{\mathbb{R}} = W_s^{\mathbb{P}} + \int_0^s \mathbf{1}_{\{u \geq H\}} \gamma du.$$

Substituting $W_s^{\mathbb{P}}$ from the above equation into equation (16) gives the asset price process under the \mathbb{R} measure as follows:

$$\frac{dS_s}{S_s} = (r + \sigma \gamma \mathbf{1}_{\{s < H\}}) ds + \sigma dW_s^{\mathbb{R}} \quad (18)$$

$$= (\mu \mathbf{1}_{\{s < H\}} + r \mathbf{1}_{\{s \geq H\}}) ds + \sigma dW_s^{\mathbb{R}}, \quad (18a)$$

where $\mathbf{1}_{\{\cdot\}}$ is an indicator function which equals 1 if the condition is satisfied (and 0 otherwise).

A highly intuitive property of the \mathbb{R} measure is that any stochastic process under this measure can be obtained by a simple inspection of that process under the \mathbb{P} measure and the \mathbb{Q} measure as follows: The stochastic process under the

\mathbb{R} measure is the physical stochastic process *until before the horizon H* , and it becomes the risk-neutral process *on or after the horizon H* . This property follows from Propositions 1(i) and 1(ii) and is also consistent with conditions C1 and C2 given earlier. Thus, the asset price process in equation (18a) (which equals equation (18) since $\mu = r + \gamma\sigma$) can be obtained by a simple inspection of the asset price process under the physical measure and the risk-neutral measure, given by equations (16) and (17), respectively. This insight is not limited to only the Black-Scholes model and holds more generally under any contingent claim model that uses the \mathbb{Q} measure for valuation. This insight is also consistent with the triangular relationship between the Brownian motions $W^{\mathbb{R}}$, $W^{\mathbb{Q}}$, and $W^{\mathbb{P}}$, given as follows:

$$W_s^{\mathbb{R}} = \begin{cases} \int_0^H dW_u^{\mathbb{P}} + \int_H^s dW_u^{\mathbb{Q}}, & \text{if } H \leq s \leq T, \\ \int_0^s dW_u^{\mathbb{P}}, & \text{if } 0 \leq s < H. \end{cases}$$

The Brownian motion under the \mathbb{R} measure is the same as the Brownian motion under the \mathbb{P} measure from time 0 until time H , and then from time H onwards, it evolves as the Brownian motion under the \mathbb{Q} measure.

To demonstrate the usefulness of the \mathbb{R} measure, consider a European call option C on the asset, maturing at time T with strike price K . Substituting $F_T = C_T = \max(S_T - K, 0) = (S_T - K)^+$ as the terminal payoff from the European call option, equation (15) in Corollary 1 gives

$$\mathbb{E}_t[C_H] = \mathbb{E}_t^{\mathbb{R}} [e^{-r(T-H)}(S_T - K)^+], \quad (19)$$

where C_H is the price of the call option at time H . Since equation (18a) implies that S_T is lognormally distributed under the \mathbb{R} measure, the expected call price can be solved easily and is given in equation (C1) in Appendix C. While Rubinstein (1984) first derived the expected price formula under the Black-Scholes model by using the property of double integrals of normally distributed variables, our

derivation is much simpler.¹²

The Black-Scholes call price formula has also been used as a part of the solution of the option valuation models with more complex stochastic processes and/or terminal boundary conditions. For example, Merton (1976) solves the call option price under a jump diffusion model using a weighted sum of an infinite number of Black-Scholes call prices, and Rubinstein (1991) solves the price of a forward-start call option as a constant share of the underlying asset's price, where the constant is determined using the Black-Scholes call price. These frameworks can be extended for computing the analytical solution of the expected future call price under these complex option valuation models by using the Black-Scholes expected future call price given by equations (19) and (C1), *as a part of their solutions*. We demonstrate this insight in Appendix D by extending Rubinstein's (1991) framework to obtain the expected future price of a forward-start call option using the \mathbb{R} measure.

B. The Affine Term Structure Models

Surprisingly, the analytical solution of the expected future price of a T -maturity pure discount bond at time H (for all $t \leq H \leq T$), has not been derived under any of the dynamic term structure models, such as the affine models of Dai and Singleton (2000) and Collin-Dufresne et al. (2008), the quadratic model of Ahn et al. (2002), and the forward-rate models of Heath et al. (1992). Using the maximal affine term structure models of Dai and Singleton (2000) as an expositional example, this section shows how such an expectation can be solved under the dynamic term structure models using the \mathbb{R} measure. Substituting $F_T = P(T, T) = 1$ in equation (15) of Corollary 1, the time t expectation of the future price of the pure

¹²Rubinstein (1984) uses the following property of normal distribution:

$$\int_{-\infty}^{\infty} \mathcal{N}(A + Bz) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \mathcal{N}\left(\frac{A}{\sqrt{1 + B^2}}\right).$$

This double integral property is hard to extend to other distributions.

discount bond at time H can be given under the \mathbb{R} measure as follows:

$$\mathbb{E}_t [P(H, T)] = \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T r_u du \right) \right]. \quad (20)$$

The above expectation can be generally solved under all dynamic term structure models that admit an analytical solution to the bond's current price,¹³ and is mathematically equivalent to the iterated expectation in equation (6), with the money market account as the numeraire, given as follows:

$$\mathbb{E}_t [P(H, T)] = \mathbb{E}_t^{\mathbb{P}} \left[\mathbb{E}_H^{\mathbb{Q}} \left[\exp \left(- \int_H^T r_u du \right) \right] \right]. \quad (21)$$

However, instead of using the above equation to obtain expected (simple) return, empirical researchers in finance have generally used expected *log* returns based upon the following equation:¹⁴

$$\mathbb{E}_t [\ln P(H, T)] = \mathbb{E}_t^{\mathbb{P}} \left[\ln \left(\mathbb{E}_H^{\mathbb{Q}} \left[\exp \left(- \int_H^T r_u du \right) \right] \right) \right]. \quad (22)$$

Analytical solutions of the *exponentially* affine and *exponentially* quadratic form exist for the inside expectation term $\mathbb{E}_H^{\mathbb{Q}} \left[\exp \left(- \int_H^T r_u du \right) \right]$ in the above equation for the affine and the quadratic term structure models, respectively. Taking the *log* of the exponential-form solution of this inside expectation term simplifies equation (22) considerably since $\log(\exp(x)) = x$. Moreover, for different econometric reasons, researchers have used expected log returns based upon equation (22), instead of using expected simple returns based upon equation (21) for studying default-free bond returns. For example, Dai and Singleton (2002), Bansal and Zhou (2002), Cochrane and Piazzesi (2005), and Eraker et al. (2015), all use $\mathbb{E}_t^{\mathbb{P}} [\ln (P(H, T)/P(t, T))]$ for the models studied in their papers.

¹³The only condition required is that the functional forms of the market prices of risks should allow maintaining the similar admissible forms under both the \mathbb{P} and \mathbb{Q} measures, so that the admissible affine or quadratic form will also hold under the \mathbb{R} measure (which is a hybrid of \mathbb{P} and \mathbb{Q} measures as shown by Proposition 1).

¹⁴Expected log return is defined as $\mathbb{E}_t [\ln (P(H, T)/P(t, T))]$, and expected simple return is defined as $\mathbb{E}_t [P(H, T) - P(t, T)]/P(t, T)$, where $P(t, T)$ is the current price.

We would like to make two important observations in the context of the above discussion. First, an analytical solution of $\mathbb{E}_t [P(H, T)]$ given by equation (20), can be generally derived using the \mathbb{R} measure for any dynamic term structure model which admits an analytical solution for the current price of the zero-coupon bond using the \mathbb{Q} measure. Second, it is generally not possible to know the analytical solution of $\mathbb{E}_t [P(H, T)]$ by knowing the analytical solution of $\mathbb{E}_t [\ln (P(H, T))]$, or vice versa, for most term structure models except over the infinitesimal horizon $H = t + dt$.¹⁵ Thus, even though the estimates of the expected log returns over a finite horizon have been obtained by many researchers (usually using one month returns), such estimates cannot be used by fixed income investors to compute the *expected returns* of bonds and bond portfolios over a finite horizon, without making additional return distribution assumptions.

In order to derive the expected returns of default-free bonds, the following derives the expected future price of a pure discount bond at any finite horizon H (for all $t \leq H \leq T$) using the affine term structure models (ATSMs) of Dai and Singleton (2000) and Collin-Dufresne et al. (2008). Collin-Dufresne et al. (2008) present a new method to obtain maximal ATSMs that uses infinitesimal maturity yields and their quadratic covariations as the globally identifiable state variables. The new method allows model-independent estimates for the state vector that can be estimated directly from yield curve data. When using three or less state variables—which is typical in empirical research on term structure models—the new approach of Collin-Dufresne et al. (2008) is consistent with the general framework of Dai and Singleton (2000) given below, but with some clear advantages in the estimation and interpretation of the economic state variables.¹⁶ This

¹⁵Ito's lemma can be used to obtain $\mathbb{E}_t [P(H, T)]$ from $\mathbb{E}_t [\ln (P(H, T))]$, or vice-versa, when $H = t + dt$, where dt is an infinitesimally small time interval.

¹⁶When using more than three state variables, the maximal affine models of Collin-Dufresne et al. (2008) can have more identifiable parameters than the “maximal” model of Dai and Singleton (2000). For these cases, the framework presented in this paper can be generalized so that the generalized $A_m(N)$ class of maximal affine models can have m square-root processes and $N - m$ Gaussian state variables, but with M number of Brownian motions, such that $N \leq M$. Extending the Dai and Singleton (2000) model so that the number of Brownian motions may exceed the number of state variables ensures that the extended model nests the more general

framework is also consistent with the unspanned stochastic volatility-based affine term structure models of Collin-Dufresne et al. (2009) with appropriate parameter restriction on the stochastic processes.

Assume that the instantaneous short rate r is an affine function of a vector of N state variables $Y = (Y_1, Y_2, \dots, Y_N)$ (which may be observed as in Collin-Dufresne et al. (2008) or latent as in Dai and Singleton (2000)), given as

$$r_s = \delta_0 + \sum_{i=1}^N \delta_i Y_{is} \triangleq \delta_0 + \delta_y' Y_s,$$

where Y follows an affine diffusion under the physical measure \mathbb{P} , as follows:

$$dY_s = \mathcal{K} (\Theta - Y_s) ds + \Sigma \sqrt{V_s} dW_s^{\mathbb{P}}. \quad (23)$$

$W^{\mathbb{P}}$ is a vector of N independent standard Brownian motions under \mathbb{P} , Θ is $N \times 1$ vector, \mathcal{K} and Σ are $N \times N$ matrices, which may be nondiagonal and asymmetric, and V is a diagonal matrix with the i th diagonal element given by

$$[V_s]_{ii} = \alpha_i + \beta_i' Y_s.$$

Assume that the market prices of risks, γ_s , are given by

$$\gamma_s = \sqrt{V_s} \gamma,$$

where γ is an $N \times 1$ vector of constants. Using the change of measure, the risk-neutral process for Y is given as,

$$dY_s = [\mathcal{K} (\Theta - Y_s) - \Sigma V_s \gamma] ds + \Sigma \sqrt{V_s} dW_s^{\mathbb{Q}} \quad (24)$$

$$= \mathcal{K}^* (\Theta^* - Y_s) ds + \Sigma \sqrt{V_s} dW_s^{\mathbb{Q}}, \quad (24a)$$

maximal affine models of Collin-Dufresne et al. (2008) when the number of state variables is more than three. For example, Collin-Dufresne et al. (2008) present a maximal $A_2(4)$ model which requires 5 Brownian motions in order to be consistent with the Dai and Singleton (2000) framework. Also, see the discussion in Collin-Dufresne et al. (2008, pp. 764-765, and footnote 19).

where $\mathcal{K}^* = \mathcal{K} + \Sigma\Phi$, $\Theta^* = \mathcal{K}^{*-1}(\mathcal{K}\Theta - \Sigma\psi)$, the i th row of Φ is given by $\gamma_i\beta_i'$, and ψ is an $N \times 1$ vector whose i th element is given by $\gamma_i\alpha_i$.

Using Lemma 1 and Internet Appendix Section I.C, the stochastic process for Y under the \mathbb{R} measure is given as follows:

$$dY_s = [\mathcal{K}(\Theta - Y_s) - \mathbf{1}_{\{s \geq H\}}\Sigma V_s \gamma] ds + \Sigma\sqrt{V_s}dW_s^{\mathbb{R}} \quad (25)$$

$$= [\mathcal{K}(\Theta - Y_s)\mathbf{1}_{\{s < H\}} + \mathcal{K}^*(\Theta^* - Y_s)\mathbf{1}_{\{s \geq H\}}] ds + \Sigma\sqrt{V_s}dW_s^{\mathbb{R}}. \quad (25a)$$

Recall from the previous section that any stochastic process under the \mathbb{R} measure is the physical stochastic process until before the horizon H , and it becomes the risk-neutral process on or after the horizon H . Thus, the state variable process under the \mathbb{R} measure in equation (25a) can also be obtained by a simple inspection of the state variable process under the \mathbb{P} measure given by equation (23), and under the \mathbb{Q} measure given by equation (24a).

As shown by Dai and Singleton (2000), all N -factor ATSMs can be uniquely classified into $N + 1$ non-nested subfamilies $\mathbb{A}_m(N)$, where $m = 0, 1, \dots, N$, indexes the degree of dependence of the conditional variances on the number of state variables Y . The $\mathbb{A}_m(N)$ model must satisfy several parametric restrictions for admissibility as outlined in Dai and Singleton (2000). Under these restrictions, the expected future price of a T -maturity pure discount bond under the $\mathbb{A}_m(N)$ model can be given as

$$\begin{aligned} \mathbb{E}_t[P(H, T)] &= \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T r_u du \right) \right] = \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T (\delta_0 + \delta_y' Y_u) du \right) \right] \\ &= \exp(-\delta_0(T - H)) \cdot \mathbb{E}_t^{\mathbb{R}} \left[\mathbb{E}_H^{\mathbb{R}} \left[\exp \left(- \int_H^T \delta_y' Y_u du \right) \right] \right]. \end{aligned} \quad (26)$$

In order to compute the above expectation, we first consider a more general expectation quantity starting at time H defined as

$$\begin{aligned} \Psi^*(Y_H; \tau) &\triangleq \mathbb{E}_H^{\mathbb{R}} \left[\exp \left(-b^{*'} Y_{H+\tau} - \int_H^{H+\tau} c^{*'} Y_u du \right) \right] \\ &= \mathbb{E}_H^{\mathbb{Q}} \left[\exp \left(-b^{*'} Y_{H+\tau} - \int_H^{H+\tau} c^{*'} Y_u du \right) \right]. \end{aligned}$$

For any well-behaved $N \times 1$ vectors b^* , c^* , Θ^* and $N \times N$ matrix \mathcal{K}^* , according to the Feynman-Kac theorem, the expected value $\Psi^*(Y_H; \tau)$ should fulfill the following PDE:

$$-\frac{\partial \Psi^*}{\partial \tau} + \frac{\partial \Psi^*}{\partial Y} \mathcal{K}^* (\Theta^* - Y) + \frac{1}{2} \text{tr} \left[\frac{\partial^2 \Psi^*}{\partial Y^2} \Sigma V \Sigma' \right] = c^{*'} Y \Psi^*,$$

with a boundary condition $\Psi^*(Y_H; 0) = \exp(-b^{*'} Y_H)$. Then, we can easily show that the solution to this PDE is

$$\Psi^*(Y_H; \tau) = \exp \left(-A_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau) - B_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau)' Y_H \right), \quad (27)$$

where $A_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau)$ and $B_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau)$ are obtained by solving the following ordinary differential equations (ODEs):

$$\begin{aligned} \frac{dA_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau)}{d\tau} &= \Theta^{*'} \mathcal{K}^{*'} B_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau) - \frac{1}{2} \sum_{i=1}^N \left[\Sigma' B_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau) \right]_i^2 \alpha_i, \\ \frac{dB_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau)}{d\tau} &= -\mathcal{K}^{*'} B_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau) - \frac{1}{2} \sum_{i=1}^N \left[\Sigma' B_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau) \right]_i^2 \beta_i + c, \end{aligned} \quad (28)$$

with the terminal conditions $A_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(0) = 0$, $B_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(0) = b^*$. $A_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau)$ and $B_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau)$ can be solved through numerical procedures such as Runge-Kutta. Furthermore, another similar expectation quantity $\Psi(Y_t; \tau)$ starting at time t with $\tau < H - t$ satisfies

$$\begin{aligned} \Psi(Y_t; \tau) &\triangleq \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(-b' Y_{t+\tau} - \int_t^{t+\tau} c' Y_u du \right) \right] = \mathbb{E}_t^{\mathbb{P}} \left[\exp \left(-b' Y_{t+\tau} - \int_t^{t+\tau} c' Y_u du \right) \right] \\ &= \exp \left(-A_{\mathcal{K}, \Theta}^{(b, c)}(\tau) - B_{\mathcal{K}, \Theta}^{(b, c)}(\tau)' Y_t \right), \end{aligned} \quad (29)$$

where $A_{\mathcal{K}, \Theta}^{(b, c)}(\tau)$ and $B_{\mathcal{K}, \Theta}^{(b, c)}(\tau)$ are obtained similarly as in equation (28) with the parameters $(b^*, c^*, \mathcal{K}^*, \Theta^*)$ replaced with parameters $(b, c, \mathcal{K}, \Theta)$.

Now we are ready to express the solution to the expected future price of a T -maturity pure discount bond at time H under the $\mathbb{A}_m(N)$ model as follows: From

equation (26), for all $t \leq H \leq T$,

$$\begin{aligned}
\mathbb{E}_t [P(H, T)] &= \exp(-\delta_0(T - H)) \cdot \mathbb{E}_t^{\mathbb{R}} \left[\mathbb{E}_H^{\mathbb{R}} \left[\exp \left(- \int_H^T \delta_y' Y_u du \right) \right] \right] \\
&= \exp(-\delta_0(T - H)) \cdot \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(-A_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(T - H) - B_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(T - H)' Y_H \right) \right] \\
&= \exp \left(-\delta_0(T - H) - A_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(T - H) - A_{\mathcal{K}, \Theta}^{(b, c)}(H - t) - B_{\mathcal{K}, \Theta}^{(b, c)}(H - t)' Y_t \right),
\end{aligned} \tag{30}$$

with

$$b^* = 0, \quad c^* = \delta_y, \quad b = B_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(T - H), \quad c = 0.$$

The second equality applies the result in equation (27), while the last equality applies the result in equation (29).

Equation (30) can be used to obtain the expected future price of a pure discount bond under any specific ATSMs of Dai and Singleton (2000) and Collin-Dufresne et al. (2008). We demonstrate three examples in Internet Appendix Section II. Specifically, Internet Appendix Sections II.A and II.B give the analytical solutions of the expected future price of a pure discount bond, under the Vasicek model and the CIR model, respectively. To our knowledge, these equations provide the first analytical solutions of the expected future price of a pure discount bond under the Vasicek and CIR models in the finance literature. Using a tedious derivation, Internet Appendix Section II.C gives the expected future price of the pure discount bond under the $A_{1r}(3)$ model of Dai and Singleton (2000). Furthermore, Internet Appendix Section III derives a general solution to the expected future price of a pure discount bond under the maximal quadratic term structure model (QTSM), as well as a specific solution under the QTSM3 model (see Ahn et al., 2002).

The \mathbb{R} measure can also be used to compute the expected yield (e.g., see Duffee, 2002) at a future horizon H , under the N -factor ATSMs, as follows:

$$\text{Expected Yield} = \frac{-\mathbb{E}_t [\ln P(H, T)]}{T - H}.$$

Hence, to derive the expected yield, it suffices for us to obtain the key term $\mathbb{E}_t [\ln P(H, T)]$. By simplifying equation (22) using the \mathbb{R} measure, we obtain

$\mathbb{E}_t [\ln P(H, T)]$ under the $\mathbb{A}_m(N)$ model as follows:¹⁷

$$\begin{aligned} \mathbb{E}_t [\ln P(H, T)] &= \mathbb{E}_t^{\mathbb{R}} \left[\ln \mathbb{E}_H^{\mathbb{R}} \left[\exp \left(- \int_H^T r_u du \right) \right] \right] \\ &= \mathbb{E}_t^{\mathbb{R}} \left[\ln \mathbb{E}_H^{\mathbb{R}} \left[\exp \left(- \int_H^T (\delta_0 + \delta_y' Y_u) du \right) \right] \right] \\ &= -\delta_0(T - H) - A_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(T - H) - B_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(T - H)' \mathbb{E}_t^{\mathbb{R}} [Y_H], \end{aligned}$$

with $b^* = 0$, $c^* = \delta_y$, $\mathbb{E}_t^{\mathbb{R}} [Y_H] = \Theta + e^{-\mathcal{K}(H-t)} (Y_t - \Theta)$, and $A_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau)$ and $B_{\mathcal{K}^*, \Theta^*}^{(b^*, c^*)}(\tau)$ are defined as in equation (28).

C. The R -Transforms

In two path-breaking papers, Bakshi and Madan (2000) and Duffie et al. (2000) derive new transforms that extend the Fourier transform-based method of Heston (1993) for the valuation of a wide variety of contingent claims. Though Chacko and Das (2002) also independently develop a generalized Fourier transform-based method for the valuation of a variety of interest rate derivatives, their approach can be obtained as a special case of the more general transform-based method of Duffie et al. (2000). Since the specific transforms in these papers use the risk-neutral measure \mathbb{Q} to do valuation, we refer to these transforms as Q -transforms.

This section derives new R -transforms using the \mathbb{R} measure to obtain the expected future prices of a wide variety of contingent claims that are priced using the Q -transforms of Bakshi and Madan (2000), Duffie et al. (2000), and Chacko and Das (2002). The R -transforms nest the corresponding Q -transforms for the special case when $H = t$. The R -transforms can be used to derive analytical solutions of the expected future prices of contingent claims that are priced under the following models: the affine option pricing models of Heston (1993), Bates (1996, 2000), Bakshi et al. (1997), Pan (2002), Bakshi and Madan (2000), Duffie et al. (2000), and Chacko and Das (2002); the Lévy option pricing models of Carr et al. (2002) and Carr and Wu (2003); the forward-start option model of Kruse and

¹⁷By applying Proposition 1, $\mathbb{E}_t^{\mathbb{R}} \left[f \left(\mathbb{E}_H^{\mathbb{R}} \left[G_H \frac{F_T}{G_T} \right] \right) \right] = \mathbb{E}_t^{\mathbb{P}} \left[f \left(\mathbb{E}_H^{\mathbb{Q}^*} \left[G_H \frac{F_T}{G_T} \right] \right) \right]$ also holds for a function $f(x)$ of x . Here, we employ the log function $f(x) = \ln(x)$, use $\mathbb{Q}^* = \mathbb{Q}$, and assume the numeraire G is the money market account.

Nögel (2005); the bond option pricing models in the maximal affine classes by Dai and Singleton (2000, 2002), Collin-Dufresne et al. (2008), Collin-Dufresne et al. (2009), and in the maximal quadratic class by Ahn et al. (2002), among others. Nawalkha and Zhuo (2022) obtain the analytical solutions of the expected future prices of equity options and a variety of interest rate derivatives using some of the above models.

As in Duffie et al. (2000), we define an affine jump diffusion state process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and assume that Y is a Markov process in some state space $D \in \mathcal{R}^N$, solving the stochastic differential equation

$$dY_s = \mu(Y_s)ds + \sigma(Y_s)dW_s^{\mathbb{P}} + dJ_s,$$

where $W^{\mathbb{P}}$ is a standard Brownian motion in \mathcal{R}^N ; $\mu : D \rightarrow \mathcal{R}^N$, $\sigma : D \rightarrow \mathcal{R}^{N \times N}$, and J is a pure jump process whose jumps have a fixed probability distribution ν on \mathcal{R}^N and arrive with intensity $\{\lambda(Y_s) : s \geq 0\}$, for some $\lambda : D \rightarrow [0, \infty)$. The transition semi-group of the process Y has an infinitesimal generator \mathcal{D} of the Lévy type, defined at a bounded \mathcal{C}^2 function $f : D \rightarrow \mathcal{R}$, with bounded first and second derivatives, by

$$\mathcal{D}f(y) = f_y(y)\mu(y) + \frac{1}{2}\text{tr}[f_{yy}(y)\sigma(y)\sigma(y)'] + \lambda(y) \int_{\mathcal{R}^N} [f(y+j) - f(y)] d\nu(j).$$

As in Duffie et al. (2000), Duffie and Kan (1996), and Dai and Singleton (2000), we assume that $(D, \mu, \sigma, \lambda, \nu)$ satisfies the joint restrictions, such that Y is well defined, and the affine dependence of μ , $\sigma\sigma'$, λ , and r are determined by coefficients (k, h, l, ρ) defined by

- $\mu(y) = k_0 + k_1y$, for $k = (k_0, k_1) \in \mathcal{R}^N \times \mathcal{R}^{N \times N}$.
- $(\sigma(y)\sigma(y)')_{ij} = (h_0)_{ij} + (h_1)_{ij} \cdot y$, for $h = (h_0, h_1) \in \mathcal{R}^{N \times N} \times \mathcal{R}^{N \times N \times N}$.
- $\lambda(y) = l_0 + l_1'y$, for $l = (l_0, l_1) \in \mathcal{R} \times \mathcal{R}^N$.
- $r(y) = \rho_0 + \rho_1'y$, for $\rho = (\rho_0, \rho_1) \in \mathcal{R} \times \mathcal{R}^N$.

For $x \in \mathcal{C}^N$, the set of N -tuples of complex numbers, we define $\theta(x) = \int_{\mathcal{R}^N} \exp(x'j) d\nu(j)$ whenever the integral is well defined. Hence, the “coefficients”

(k, h, l, θ) capture the distribution of Y , and a “characteristic” $\chi = (k, h, l, \theta, \rho)$ captures both the distribution of Y as well as the effects of any discounting. Furthermore, we assume the state vector Y to be an affine jump diffusion with coefficients $(k^*, h^*, l^*, \theta^*)$ under the risk-neutral measure \mathbb{Q} , and the relevant characteristic for risk-neutral pricing is then $\chi^* = (k^*, h^*, l^*, \theta^*, \rho^*)$.

Duffie et al. (2000) define a transform $\phi : \mathcal{C}^N \times \mathcal{R}_+ \times \mathcal{R}_+ \rightarrow \mathcal{C}$ of Y_T conditional on \mathcal{F}_t , when well defined for all $t \leq T$, as¹⁸

$$\phi(z; t, T) \triangleq \mathbb{E}_t^{\mathbb{Q}} \left[\exp \left(- \int_t^T r(Y_u) du \right) \exp(z'Y_T) \right]. \quad (31)$$

The current prices of various types of contingent claims can be computed conveniently using the above transform. Since this transform is derived to value the current price of contingent claims, we refer to it as the Q -transform. To extend the Q -transform and compute the expected future prices of contingent claims, we define an R -transform $\phi^R : \mathcal{C}^N \times \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{R}_+ \rightarrow \mathcal{C}$ of Y_T conditional on \mathcal{F}_t , when well defined for all $t \leq H \leq T$, as

$$\phi^R(z; t, T, H) \triangleq \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T r(Y_u) du \right) \exp(z'Y_T) \right]. \quad (32)$$

Proposition A1 in Appendix E obtains the solution of the R -transform ϕ^R . For the special case of $H = t$, the R -transform and its solution given by equations (32) and (E1), reduce to the Q -transform and its solution, respectively in Duffie et al. (2000, equations 2.3 and 2.4). Nawalkha and Zhuo (2022) use the R -transform to derive the analytical solutions of the expected future prices of equity options under the stochastic-volatility-double-jump (SVJJ) model of Duffie et al. (2000) and the CGMY model of Carr et al. (2002), and interest rate derivatives under the multifactor affine jump diffusion model of Chacko and Das (2002). As an extension of the R -transform given in equation (32), Internet Appendix Section IV proposes a new *forward-start R-transform*, which can be used to obtain the expected future

¹⁸Notably, when $z = iu$ (where $i = \sqrt{-1}$), this $\phi(z; t, T)$ becomes the Q -transform of Bakshi and Madan (2000) (see equation (6)), defined as the characteristic function of the state-price density.

price of a forward-start option under affine-jump diffusion processes.

While the Q -transform given in equation (31) is sufficient for the valuation of many contingent claims—for example, standard equity options, discount bond options, caps and floors, exchange rate options, chooser options, digital options, etc.—certain pricing problems like Asian option valuation or default-time distributions are easier to solve using the extended transform given by equation (2.13) in Duffie et al. (2000). The following proposes an *extended R*-transform corresponding to their extended transform, which we refer to as the *extended Q*-transform. We define the extended R -transform $\varphi^R : \mathcal{R}^N \times \mathcal{C}^N \times \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{R}_+ \rightarrow \mathcal{C}$ of Y_T conditional on \mathcal{F}_t , when well defined for $t \leq H \leq T$, as

$$\varphi^R(v, z; t, T, H) \triangleq \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T r(Y_u) du \right) (v'Y_T) \exp(z'Y_T) \right]. \quad (33)$$

Proposition A2 in Appendix E gives the solution to the extended R -transform φ^R . For the special case of $H = t$, the extended R -transform and its solution given by equations (33) and (E3) reduce to the extended Q -transform and its solution, respectively in Duffie et al. (2000, equations 2.13 and 2.14).¹⁹

III. The \mathbb{R}_1^T Measure

This section considers different applications of the \mathbb{R}_1^T measure, which is perhaps the next most important EEM after the \mathbb{R} measure, and is derived as a generalization of the forward measure \mathbb{Q}^T . Assuming similar forms for the stochastic processes under the physical measure \mathbb{P} and the forward measure \mathbb{Q}^T , the \mathbb{R}_1^T measure leads to analytical solutions of expected future prices of contingent claims under all models that admit an analytical solution to the claim's price using the forward measure. The following corollary to Theorem 2 can be used for all applications of the \mathbb{R}_1^T measure.

COROLLARY 2: *When the numeraire asset is a T -maturity pure discount bond, i.e., $G = P(\cdot, T)$, then $\mathbb{R}_1^* = \mathbb{R}_1^T$, and the expected future price of the contingent*

¹⁹An application of the extended R -transform for the derivation of the analytical solution of the expected future price of an Asian option on interest rates is given by Nawalkha and Zhuo (2022).

claim G , can be given using equation (11) as

$$\mathbb{E}_t [F_H] = \mathbb{E}_t^{\mathbb{P}} [P(H, T)] \mathbb{E}_t^{\mathbb{R}_1^T} [F_T]. \quad (34)$$

Let S_T be the terminal value of the asset underlying the claim, and $V = S/P(\cdot, T)$ define the normalized asset price. Then the expectation of V_T under the \mathbb{R}_1^T measure can be computed as follows:

$$\mathbb{E}_t^{\mathbb{R}_1^T} [V_T] = \frac{\mathbb{E}_t^{\mathbb{P}} [S_H]}{\mathbb{E}_t^{\mathbb{P}} [P(H, T)]}. \quad (35)$$

Proof. Equations (34) and (35) are direct applications of equations (11) and (14), respectively, with $G = P(\cdot, T)$ and $P(T, T) = 1$. \square

The \mathbb{R}_1^T measure can be used to derive the expected future prices of contingent claims under the following models:²⁰ the stochastic interest rate-based equity option pricing model of Merton (1973b) and its extensions; the stochastic interest rate-based corporate debt pricing models of Longstaff and Schwartz (1995), Jarrow et al. (1997) and Collin-Dufresne and Goldstein (2001); various term structure models for pricing bond options and caps, such as Dai and Singleton (2000, 2002), Collin-Dufresne et al. (2008); Ahn et al. (2002), Leippold and Wu (2003); Heath et al. (1992), Miltersen et al. (1997), Brace et al. (1997), and Jamshidian (1997); and the currency option pricing models of Grabbe (1983), Amin and Jarrow (1991), and Hilliard et al. (1991), among others.

We present the \mathbb{R}_1^T measure using the examples of the Merton (1973b) option pricing model and the Collin-Dufresne and Goldstein (2001) corporate debt pricing model in Sections III.A and III.B, respectively. Section III.C extends the theoretical framework of Breeden and Litzenberger (1978) to obtain the \mathbb{R}_1^T density and the physical density of the underlying asset using non-parametric and semi-parametric option return models.

²⁰We apply the \mathbb{R}_1^T measures to some of the models in the main part of the paper, and provide this measure under a multidimensional Brownian motion in Internet Appendix Section I.D, which can also be applied to other models not explicitly considered in the main part of the paper.

A. The Merton Model

As in Merton (1973b), we assume the asset price S is described by

$$\frac{dS_s}{S_s} = \mu(s)ds + \sigma(s)dW_{1s}^{\mathbb{P}}, \quad (36)$$

where $W_{1s}^{\mathbb{P}}$ is the Brownian motion under \mathbb{P} , $\mu(s)$ is the instantaneous expected return which may be stochastic, and $\sigma(s)$ is the instantaneous volatility which is assumed to be deterministic.

For a T -maturity pure discount bond $P(\cdot, T)$, we assume its price follows

$$\frac{dP(s, T)}{P(s, T)} = \mu_P(s, T)ds + \sigma_P(s, T)dW_{2s}^{\mathbb{P}}, \quad (37)$$

where $W_{2s}^{\mathbb{P}}$ is the Brownian motion associated with the T -maturity pure discount bond, which has a deterministic correlation with $W_{1s}^{\mathbb{P}}$, given by the coefficient $\rho(s)$; $\mu_P(s, T)$ is the instantaneous expected return, which may be stochastic; and $\sigma_P(s, T)$ is the instantaneous volatility which is assumed to be deterministic.

Consider a European call option C written on the asset S with a strike price of K , and an option expiration date equal to T . Using the T -maturity pure discount bond as the numeraire, the expected future price of this option is given by equation (34) in Corollary 2, with $F_T = C_T = (S_T - K)^+$ as the terminal payoff given as follows:

$$\begin{aligned} \mathbb{E}_t[C_H] &= \mathbb{E}_t^{\mathbb{P}} [P(H, T)] \mathbb{E}_t^{\mathbb{R}_1^T} [(S_T - K)^+] \\ &= \mathbb{E}_t^{\mathbb{P}} [P(H, T)] \mathbb{E}_t^{\mathbb{R}_1^T} \left[\frac{(S_T - K)^+}{P(T, T)} \right] \\ &= \mathbb{E}_t^{\mathbb{P}} [P(H, T)] \mathbb{E}_t^{\mathbb{R}_1^T} [(V_T - K)^+], \end{aligned} \quad (38)$$

where $V = S/P(\cdot, T)$ is the asset price normalized by the numeraire.

²¹While Merton (1973b, equation (25)) assumes a constant correlation, we allow the asset return's correlation with the bond return to change *deterministically* as the bond maturity declines over time. Doing this also allows *bond option models* based on the Gaussian multifactor term structure models to be nested in the framework presented in this section by assuming that the asset underlying the option contract is a pure discount bond maturing at any time after T .

The derivation of an analytical solution of the expected future call price (of a similar form as in Merton, 1973b) using equation (38) requires that the asset price normalized by the bond price be distributed lognormally under the \mathbb{R}_1^T measure. Based upon Theorem 2, and equations (36) and (37), *sufficient* assumptions for this to occur are:

- i) the physical drift of the asset price process is of the form $\mu(s) = r_s + \gamma(s)$, where the risk premium $\gamma(s)$ is deterministic, and
- ii) the short rate process r_s , and the bond price process in equation (37) are consistent with the various multifactor Gaussian term structure models (see Dai and Singleton, 2000; Heath et al., 1992), such that the physical drift of the bond price process is of the form $\mu_P(s, T) = r_s + \gamma_P(s, T)$, where the risk premium $\gamma_P(s, T)$ is deterministic.

Without specifying a specific Gaussian term structure model for the short rate process it is not possible to express the asset price process and the bond price process under the \mathbb{R}_1^T measure. However, even without knowing the term structure model, the two assumptions given above allow the derivation of the analytical solution of the expected future call price, which is very similar in its form to the Merton's (1973b) original formula. Our solution also generalizes Merton's formula to hold under multifactor Gaussian term structure models for the special case of $H = t$.²²

Assume that based upon equations (36) and (37), $\ln V_T$ is normally distributed under both the \mathbb{P} measure and the \mathbb{Q}^T measure with the same variance equal to v_p^2 .²³ Using the Radon-Nikodým derivative process of \mathbb{R}_1^T with respect to \mathbb{P} given in Lemma 2, and the two assumptions given above, $\ln V_T$ is also distributed normally under the \mathbb{R}_1^T measure with the same variance v_p^2 . The change of measure from \mathbb{P} to

²²Merton's (1973b, footnote 43) original model assumes a constant (and not deterministic) correlation between the asset price process and the bond price process, which is consistent with only single-factor Gaussian term structure models.

²³It is well-known that $V = S/P(\cdot, T)$ is a martingale under the \mathbb{Q}^T measure. This implicitly follows from Merton's (1973b) PDE using Feynman Kac theorem, and explicitly by applying the \mathbb{Q}^T measure as in Geman et al. (1995). Since the change of measure from \mathbb{P} to \mathbb{Q}^T changes only the deterministic drift, the variance of $\ln V_T$ remains the same under both measures.

either \mathbb{Q}^T or to \mathbb{R}_1^T , changes the drift of the $\ln V$ process from a given deterministic drift to another deterministic drift, which does not change the variance of $\ln V_T$.²⁴ Thus, we can express the expectation of $\ln V_T$ under the \mathbb{R}_1^T measure as follows:

$$\mathbb{E}_t^{\mathbb{R}_1^T} [\ln V_T] = \ln \mathbb{E}_t^{\mathbb{R}_1^T} [V_T] - \frac{1}{2} \text{Var}_t^{\mathbb{R}_1^T} [\ln V_T] = \ln \frac{\mathbb{E}_t^{\mathbb{P}} [S_H]}{\mathbb{E}_t^{\mathbb{P}} [P(H, T)]} - \frac{1}{2} v_p^2, \quad (39)$$

where the second equality follows from equation (35) of Corollary 2.

Simplifying equation (38), gives

$$\begin{aligned} \mathbb{E}_t [C_H] &= \mathbb{E}_t^{\mathbb{P}} [P(H, T)] \mathbb{E}_t^{\mathbb{R}_1^T} [V_T \mathbf{1}_{\{\ln V_T > \ln K\}}] - K \mathbb{E}_t^{\mathbb{P}} [P(H, T)] \mathbb{E}_t^{\mathbb{R}_1^T} [\mathbf{1}_{\{\ln V_T > \ln K\}}] \\ &= \mathbb{E}_t^{\mathbb{P}} [P(H, T)] \mathbb{E}_t^{\mathbb{R}_1^T} [V_T] \mathbb{E}_t^{\hat{\mathbb{R}}_1^T} [\mathbf{1}_{\{\ln V_T > \ln K\}}] - K \mathbb{E}_t^{\mathbb{P}} [P(H, T)] \mathbb{E}_t^{\mathbb{R}_1^T} [\mathbf{1}_{\{\ln V_T > \ln K\}}] \\ &= \mathbb{E}_t^{\mathbb{P}} [S_H] \mathbb{E}_t^{\hat{\mathbb{R}}_1^T} [\mathbf{1}_{\{\ln V_T > \ln K\}}] - K \mathbb{E}_t^{\mathbb{P}} [P(H, T)] \mathbb{E}_t^{\mathbb{R}_1^T} [\mathbf{1}_{\{\ln V_T > \ln K\}}], \end{aligned}$$

where $\hat{\mathbb{R}}_1^T$ is an adjusted probability measure defined by $\mathbb{E}_t^{\hat{\mathbb{R}}_1^T} [Z_T] = \mathbb{E}_t^{\mathbb{R}_1^T} \left[\left(V_T / \mathbb{E}_t^{\mathbb{R}_1^T} [V_T] \right) \cdot Z_T \right]$ for a given \mathcal{F}_T -measurable variable Z_T , and the last equality follows since $\mathbb{E}_t^{\mathbb{P}} [P(H, T)] \mathbb{E}_t^{\mathbb{R}_1^T} [V_T] = \mathbb{E}_t^{\mathbb{P}} [S_H]$ using equation (35). The solution to the expected future call price follows from the above equation, and is given as

$$\mathbb{E}_t [C_H] = \mathbb{E}_t [S_H] \mathcal{N}(\hat{d}_1) - K \mathbb{E}_t [P(H, T)] \mathcal{N}(\hat{d}_2), \quad (40)$$

where

$$\begin{aligned} \hat{d}_1 &= \frac{1}{v_p} \ln \frac{\mathbb{E}_t [S_H]}{\mathbb{E}_t [P(H, T)] K} + \frac{v_p}{2}, & \hat{d}_2 &= \frac{1}{v_p} \ln \frac{\mathbb{E}_t [S_H]}{\mathbb{E}_t [P(H, T)] K} - \frac{v_p}{2}, \\ v_p &= \sqrt{\int_t^T [\sigma(u)^2 - 2\rho(u)\sigma(u)\sigma_P(u, T) + \sigma_P(u, T)^2] du}. \end{aligned}$$

The above analytical solution depends on four variables: $\mathbb{E}_t [S_H]$, $\mathbb{E}_t [P(H, T)]$, v_p , and K (the option expiration, $T - t$, is implicitly included in $\mathbb{E}_t [P(H, T)]$ and v_p), and reduces to the Rubinstein's (1984) formula given in Appendix C (see

²⁴As an example, see the case of the Merton model with the single-factor Vasicek model in Internet Appendix Section V.

equation (C1)), assuming a constant asset return volatility, a constant asset drift, and a constant short rate. The analytical solution also nests the Merton's (1973b) solution for the current call price for the special case of $H = t$, under which the \mathbb{R}_1^T measure becomes the \mathbb{Q}^T forward measure (see Geman, 1989), as well as the solution of the expected future price of a call option written on a pure discount bond under all multifactor Gaussian term structure models.²⁵

Under the two assumptions given above, the definition of v_p remains the same as under the Merton's (1973b) model, and the specific solutions of $\mathbb{E}_t[S_H]$ and $\mathbb{E}_t[P(H, T)]$ can be derived with the appropriate specifications of the asset price process and the bond price process in equations (36) and (37), respectively. These solutions can be shown to be consistent with various multifactor asset pricing models, such as the Merton's (1973a) ICAPM or the APT (see Ross, 1976; Connor and Korajczyk, 1989), which allow stochastic interest rates but with the additional restrictions of non-stochastic volatilities and non-stochastic risk premiums.²⁶ For example, Internet Appendix Section V derives a full solution of the expected call price given by equation (40), which is consistent with the Merton's (1973a) 2-factor ICAPM with the Vasicek's (1977) short rate process.

The expected option price solution in equation (40) can be obtained by simply replacing the current prices S_t and $P(t, T)$ with the expected future prices $\mathbb{E}_t[S_H]$ and $\mathbb{E}_t[P(H, T)]$, respectively, everywhere in the Merton's (1973b) original formula.²⁷ Interestingly, this handy feature of the new analytical solution applies to even other option pricing models with deterministic volatilities and deterministic risk premiums for the underlying asset return processes, including: i) the currency option price models of Grabbe (1983), Amin and Jarrow (1991), and Hilliard et al. (1991), in which the expected future price of the option written on the currency

²⁵See footnote 21.

²⁶While the theoretical frameworks of Merton (1973a, equation 7) and Connor and Korajczyk (1989, equation 1) allow stochastic volatilities and stochastic risk premiums in an intertemporal setting, the principal components-based or the macrovariables-based empirical estimations of the unconditional ICAPM and the unconditional intertemporal APT assume constant factor volatilities and constant risk premiums.

²⁷Of course, we also allow a *deterministic* correlation term in the Merton's formula to allow for multiple Gaussian factors.

exchange rate S (the domestic price of 1 unit of foreign currency) can be obtained by replacing the domestic currency-based current prices of the foreign pure discount bond and the domestic pure discount bond, that is, $S_t P_f(t, T)$ and $P_d(t, T)$, with the expected future prices $\mathbb{E}_t [S_H P_f(H, T)]$ and $\mathbb{E}_t [P_d(H, T)]$, respectively, everywhere in the option price solution under these models; and ii) the [Margrabe's \(1978\)](#) exchange option model, in which the expected future price of the option to exchange asset S_1 for asset S_2 , can be obtained by replacing the current prices S_{1t} and S_{2t} with the expected future prices $\mathbb{E}_t [S_{1H}]$ and $\mathbb{E}_t [S_{2H}]$, respectively, everywhere in the [Margrabe's \(1978\)](#) option price solution (see Internet Appendix Section [VI](#)).

B. The Collin-Dufresne and Goldstein Model

This section uses the EEM \mathbb{R}_1^T to obtain the expected future price of a corporate bond under the Collin-Dufresne and Goldstein ([2001](#)) (CDG hereafter) structural model. The CDG model allows the issuing firm to continuously adjust its capital structure to maintain a stationary mean-reverting leverage ratio. The firm's asset return process and the short rate process under the CDG model are given as follows:

$$\begin{aligned} \frac{dS_s}{S_s} &= (r_s + \gamma^S \sigma) ds + \sigma \left(\rho dW_{1s}^{\mathbb{P}} + \sqrt{1 - \rho^2} dW_{2s}^{\mathbb{P}} \right), \\ dr_s &= \alpha_r (m_r - r_s) ds + \sigma_r dW_{1s}^{\mathbb{P}}, \end{aligned}$$

where $\alpha_r, m_r, \sigma_r, \sigma > 0$, $-1 \leq \rho \leq 1$, $\gamma^S \in \mathbb{R}$, $Z_1^{\mathbb{P}}$ and $Z_2^{\mathbb{P}}$ are independent Brownian motions under the physical measure \mathbb{P} . The market prices of risks associated with the two Brownian motions are given as

$$\begin{aligned} \gamma_{1s} &= \gamma_r, \\ \gamma_{2s} &= \frac{1}{\sqrt{1 - \rho^2}} (-\rho \gamma_r + \gamma^S). \end{aligned}$$

CDG assume that the face value of firm's total debt F^D follows a stationary

mean reverting process given as²⁸

$$\frac{dF_s^D}{F_s^D} = \lambda (\ln S_s - \nu - \phi r_s - \ln F_s^D) ds.$$

By defining the log-leverage ratio as $l \triangleq \ln(F^D/S)$, and using Itô's lemma, we obtain the physical process for the log-leverage ratio as

$$dl_s = \lambda \left(\bar{l} - \frac{\sigma \gamma^S}{\lambda} - \left(\frac{1}{\lambda} + \phi \right) r_s - l_s \right) ds - \sigma \left(\rho dW_{1s}^{\mathbb{P}} + \sqrt{1 - \rho^2} dW_{2s}^{\mathbb{P}} \right),$$

where $\bar{l} \triangleq \frac{\sigma^2}{2\lambda} - \nu$.

Then using Lemma 2 and Internet Appendix Section I.D, the processes of the state variables $\{l, r\}$ under the \mathbb{R}_1^T measure are given as

$$\begin{aligned} dl_s &= \lambda \left(\bar{l} - \left(\frac{1}{\lambda} + \phi \right) r_s - l_s + \frac{\rho \sigma \sigma_r}{\lambda} B_{\alpha_r}(T - s) - \left(\frac{\rho \sigma \sigma_r}{\lambda} B_{\alpha_r}(H - s) + \frac{\sigma \gamma^S}{\lambda} \right) \mathbf{1}_{\{s < H\}} \right) ds \\ &\quad - \sigma \left(\rho dW_{1s}^{\mathbb{R}_1^T} + \sqrt{1 - \rho^2} dW_{2s}^{\mathbb{R}_1^T} \right), \\ dr_s &= \alpha_r \left(m_r - \frac{\sigma_r \gamma_r}{\alpha_r} - r_s - \frac{\sigma_r^2}{\alpha_r} B_{\alpha_r}(T - s) + \left(\frac{\sigma_r^2}{\alpha_r} B_{\alpha_r}(H - s) + \frac{\sigma_r \gamma_r}{\alpha_r} \right) \mathbf{1}_{\{s < H\}} \right) ds \\ &\quad + \sigma_r dW_{1s}^{\mathbb{R}_1^T}, \end{aligned} \tag{41}$$

where $B_\alpha(\tau) = (1/\alpha)(1 - e^{-\alpha\tau})$.

Consider a risky discount bond issued by the firm with a promised payment of 1 at the bond maturity date T . Under the CDG framework, all bonds issued by the firm default simultaneously when the log-leverage ratio l hits a pre-specified constant $\ln K$ from below. In the event of default, the bondholder receives a fraction $1 - \omega$ times the present value of a default-free zero coupon bond that pays 1 at the maturity date T . Under these payoff assumptions, the current price of a risky discount bond $D(t, T)$ maturing at time T is given by Collin-Dufresne and

²⁸The Longstaff and Schwartz (1995) model obtains as a special case of the CDG model when the total debt of the firm is constant.

Goldstein (2001) as

$$D(t, T) = P(t, T) \left(1 - \omega \mathbb{E}_t^{\mathbb{Q}^T} \left[\mathbf{1}_{\{\tau_{[t, T]} < T\}} \right] \right),$$

where \mathbb{Q}^T is the forward measure, the variable $\tau_{[t, T]}$ is the first passage time to default during the time period t to T , and $\mathbb{E}_t^{\mathbb{Q}^T} \left[\mathbf{1}_{\{\tau_{[t, T]} < T\}} \right]$ is the \mathbb{Q}^T probability of the log-leverage ratio hitting the boundary $\ln K$ from below during this time period.

We derive the expected future bond price by generalizing the above equation using the \mathbb{R}_1^T measure. Substituting the terminal payoff $F_T = 1 - \omega \mathbf{1}_{\{\tau_{[t, T]} < T\}}$ in equation (34) of Corollary 2, the expected future bond price is given as

$$\mathbb{E}_t [D(H, T)] = \mathbb{E}_t^{\mathbb{P}} [P(H, T)] \left(1 - \omega \mathbb{E}_t^{\mathbb{R}_1^T} \left[\mathbf{1}_{\{\tau_{[t, T]} < T\}} \right] \right), \quad (42)$$

where $\mathbb{E}_t^{\mathbb{R}_1^T} \left[\mathbf{1}_{\{\tau_{[t, T]} < T\}} \right]$ is the \mathbb{R}_1^T probability of the log-leverage ratio hitting the boundary $\ln K$ during the time period t to T . This probability can be approximated in semi-closed form using a single numerical integral by generalizing Mueller's (2000) extension of the CDG formula for the risky bond price. The solution of the expected future bond price in equation (42) is given in Appendix F.

C. Non-Parametric and Semi-Parametric Applications

Until now we have developed the EEM theory in the context of parametric models assuming the *same form* for the state variable processes under both the physical measure and the given EMM. This assumption does not always hold empirically. For example, the physical process and the risk-neutral process for the asset return cannot have the same form under the cumulative prospect theory model of option returns given by Baele et al. (2019). This section gives additional theoretical results related to the EEM theory which do not require strong parametric assumptions.

To derive these results we assume that the expected future prices of standard European options with a sufficiently large range of strike prices have been estimated using non-parametric or semi-parametric methods (e.g., see Coval and

Shumway, 2001; Bondarenko, 2003, 2014; Jones, 2006; Driessen and Maenhout, 2007; Bakshi et al., 2010; Constantinides et al., 2013; Israelov and Kelly, 2017; Baele et al., 2019; Büchner and Kelly, 2022). Using these expected future option prices, we show how one can obtain *both* the \mathbb{R}_1^T density and the physical density of the underlying asset return using an extension of the theoretical framework of Breeden and Litzenberger (1978). Furthermore, we show that the \mathbb{R}_1^T density can be used to obtain the expected future prices (and expected returns) of a wider range of complex European contingent claims with terminal payoffs that are arbitrary functions of the underlying asset's future price.

To fix ideas assume that the Arrow-Debreu price density or the state price density (SPD) of the underlying asset at current time t is given by the function $f_t(S_T)$, where S_T is the future price of this asset at time T . Also, assume that the future state price density (FSPD) of this asset at a future time H is given by the function $f_H(S_T|Y_H)$, where Y_H is a vector of state variables (such as future asset price S_H , future volatility, etc.) at time H , for all $t \leq H \leq T$. Let the physical expectation $g_t(H, S_T) \triangleq \mathbb{E}_t[f_H(S_T|Y_H)]$ define the expected FSPD.

Consider a European call option with a payoff equal to $C_T = (S_T - K)^+$ and a European put option with a payoff equal to $P_T = (K - S_T)^+$ at time T , and let C_H and P_H be the future prices of the call option and the put option, respectively, at a future time $H \leq T$. Theorem 3 presents an extension of the Breeden and Litzenberger (1978) result, showing the relationship of the expected FSPD with the probability densities of S_T under the \mathbb{P} , \mathbb{R} and \mathbb{R}_1^T measures.

THEOREM 3: *The expected FSPD, $g_t(H, S_T) \triangleq \mathbb{E}_t[f_H(S_T|Y_H)]$, is given by the second derivative of either the expected future call price $\mathbb{E}_t[C_H]$ or the expected future put price $\mathbb{E}_t[P_H]$ with respect to the strike price K , and is related to the probability densities of S_T under the \mathbb{R}_1^T , \mathbb{R} , and \mathbb{P} measures, as follows:*

$$\begin{aligned} g_t(H, S_T) &= \left[\frac{\partial^2 \mathbb{E}_t[C_H]}{\partial K^2} \right]_{K=S_T} = \left[\frac{\partial^2 \mathbb{E}_t[P_H]}{\partial K^2} \right]_{K=S_T} \\ &= \mathbb{E}_t^{\mathbb{P}} [P(H, T)] p_t^{\mathbb{R}_1^T}(S_T), \quad \text{for all } t \leq H \leq T, \\ &= p_t^{\mathbb{P}}(S_T), \quad \text{for all } t \leq H = T, \end{aligned} \tag{43}$$

which under the special case of constant or deterministic short rate becomes,

$$\begin{aligned} g_t(H, S_T) &= \left[e^{-\int_H^T r_u du} \right] p_t^{\mathbb{R}}(S_T), \quad \text{for all } t \leq H \leq T, \\ &= p_t^{\mathbb{P}}(S_T), \quad \text{for all } t \leq H = T, \end{aligned} \quad (44)$$

where $p_t^{\mathbb{R}_1^T}(S_T)$, $p_t^{\mathbb{R}}(S_T)$, and $p_t^{\mathbb{P}}(S_T)$ are time t probability densities of S_T under \mathbb{R}_1^T , \mathbb{R} , and \mathbb{P} measures, respectively. Moreover, the expected future price of a contingent claim F with an arbitrary terminal payoff function $F_T = h(S_T)$, can be numerically computed using the expected FSPD, $g_t(H, S_T)$ as follows:

$$\mathbb{E}_t[F_H] = \int_0^\infty g_t(H, S)h(S)dS. \quad (45)$$

Proof. See Appendix A (Section E) for the proofs of equations (43) and (44). The proof of equation (45) follows by substituting $g_t(H, S_T)$ from equation (43) into equation (34), and simplifying. \square

Equation (43) shows that the second derivative of the expected future call price function with respect to the strike price gives the expected FSPD, which equals the discounted \mathbb{R}_1^T probability density, similar to how the SPD equals the discounted forward density \mathbb{Q}^T in Breeden and Litzenberger (1978). If the expected FSPD can be econometrically estimated using the expected future prices of standard European call and put options,²⁹ then it can be used to obtain the expected future prices (and expected returns) of a wider range of complex European contingent claims with terminal payoffs that are arbitrary functions of the underlying asset's future price as shown in equation (45). This can be useful if a researcher desires consistency between the expected returns of such complex claims with those of the standard European options without making strong parametric assumptions about the underlying state variable processes.

A surprisingly simple and overlooked result in the option pricing literature is given by the last equality in both equations (43) and (44), according to which

²⁹We demonstrate how the expected FSPD can be extracted using the expected future prices of standard European call and put options in Internet Appendix Section VII by extending the methodology of Jackwerth (2004).

the expected *terminal* FSPD (i.e., $g_t(T, S_T)$)—defined as the second derivative of the expected terminal call (or put) price with respect to the strike price—equals the physical density of the underlying asset’s future price. This arbitrage-based, tautologically-exact relationship between the expected terminal FSPD and the physical density is especially relevant in the light of the recent theoretical and empirical research that investigates the relationship between the SPD and the physical density (see Ross, 2015; Martin and Wagner, 2019; Jensen et al., 2019; Jackwerth and Menner, 2020).³⁰ The empirical implications of Theorem 3 can be explored by future research on non-parametric and semi-parametric option return models.

IV. Extensions and Conclusion

This section considers some extensions and applications of the EEM theory and then concludes the paper. Until now the paper has focused only on those contingent claim models that admit an analytical solution to the claim’s current price. As the first extension, consider a general claim (e.g., European, Barrier, Bermudan, quasi-American, etc.) that remains alive until time H , but the analytical solution of its current price does not exist. Numerical methods can be used to compute the expected future price of such a claim at time H using the EEMs, such as the \mathbb{R} measure. For example, consider a quasi-American put option which is exercisable on or after time $T_e \geq H$, until the option expiration date T (i.e., $T_e \leq T$). The expected future price of this option at time H is given as

$$\mathbb{E}_t [P_H^A] = \sup_{\tau \in \mathcal{T}_{T_e, T}} \mathbb{E}_t^{\mathbb{R}} \left[e^{-\int_H^\tau r_u du} P_\tau^A \right] \quad \text{for } 0 \leq t \leq H \leq T_e \leq T,$$

where P_s^A is the price of the quasi-American put option at time s , for all $H \leq s \leq T$, $P_T^A = P_T = \max(K - S_T, 0)$, \mathcal{T} is the set of all stopping times with respect

³⁰While the motivation of the Ross recovery paradigm is that only today’s option price data contains useful forward-looking information about the distribution of the future asset prices, the motivation of using Theorem 3 is that expected future option prices estimated using “recent” option price data (e.g., daily option prices from the past 3 to 6 months) contain useful forward-looking information about the distribution of future asset prices.

to the filtration \mathcal{F} , and $\mathcal{T}_{T_e, T} \triangleq \{\tau \in \mathcal{T} | \mathbb{P}(\tau \in [T_e, T]) = 1\}$ is the subset of \mathcal{T} . The above equation follows by applying Propositions 1(i) and 1(ii), and valuing the American option as an optimal stopping problem (see Jacka, 1991) from time T_e until time T . The above example shows how the EEM theory can be applied for obtaining the expected future price of any type of claim at time H , assuming the claim remains alive until time H . Numerical methods such as trees, finite difference methods, and Monte Carlo methods can be used for pricing such claims under the \mathbb{R} measure.

Second, consider those claims that may not be alive until time H in some states of nature. For example, consider a standard American option that can be exercised at any time until the expiration date of the option. While the expected future price of an American option cannot be computed, since the option is not alive at time H in some states of nature, the expected future price of a *reinvestment strategy* that invests the value of the American option upon exercise in another security—for example, an H -maturity pure discount bond or a money market account—until time H , can be computed numerically using the \mathbb{R} measure.

Third, consider those claims that pay out a stream of cash flows from time t until time T —for example, a bond that makes periodic coupon payments. The *expected returns* of such claims over the horizon H cannot be computed without knowing the *reinvestment strategy* that invests the cash flows maturing *before* time H . Like the case in the second point above, all of the cash flows maturing before time H can be reinvested in another security—for example, an H -maturity pure discount bond or a money market account—for the remaining duration until time H , and the expected future value of these reinvested cash flows can be computed numerically using the \mathbb{R} measure.

Fourth, though this paper has focused on the derivation of the analytical solutions of only the expected returns of contingent claims, future research can use the EEM theory to derive analytical solutions of the variance, covariance, and other higher-order moments of contingent claim returns.³¹ Also, various numerical methods, such as trees, finite difference methods, and Monte Carlo methods can be used to obtain the higher order moments of the return on a portfolio of claims

³¹For example, see Nawalkha and Zhuo (2022).

under the \mathbb{R} measure. Furthermore, various finite-horizon tail risk measures (such as Value-at-Risk, conditional VaR, expected shortfall, and others) for contingent claim portfolio returns can be obtained with Monte Carlo simulations under the \mathbb{R} measure. Thus, the EEMs such as the \mathbb{R} measure can significantly advance the risk and return analysis of fixed income and financial derivative portfolios.

Finally, perhaps the most important empirical application of the analytical solutions of the expected future prices of contingent claims obtained using the EEM theory is that various methods, such as Markov Chain Monte Carlo, generalized method of moments (see Hansen, 1982), and others become feasible for studying both the *cross-section* and the *term structure* of returns of a variety of contingent claims, such as Treasury bonds, corporate bonds, interest rate derivatives, credit derivatives, equity derivatives, and others. It is our hope that the EEM theory can shift the focus of research from mostly valuation to *both* valuation and expected returns, for a variety of finite-maturity securities and claims used in the fixed income markets and the derivatives markets.

Before concluding we would like to underscore an important point regarding the derivations of the expected future prices of different contingent claims under various classes of models in the equity, interest rate, and credit risk areas, given in Sections II and III, and in Internet Appendix Sections II, III, IV, V, and VI. Since the EEMs and the R -transforms provide a *single analytical solution* for the expected future price (which includes both the current price for $H = t$, and the expected future price for all $H > t$), and since a given EEM and a given R -transform always nest the associated EMM and the associated Q -transform, respectively, for the special case when $H = t$, all future work on the derivations of the analytical solutions of the expected future prices of contingent claims can use either the EEMs or the R -transforms instead of using the associated EMMs or the associated Q -transforms, respectively. Since the expected future price is the most relevant input, and in many cases the only input required for computing the expected return of a contingent claim, the use of the EEMs or the R -transforms kills two birds with one stone—it allows the derivation of a single analytical solution that does both *valuation* and computes the *expected return* of the claim.

Appendix A. The Proofs

A. Proof of Lemma 1

Proof. To prove that \mathbb{R}^* and \mathbb{P} are equivalent probability measures, and $\mathcal{L}_s^*(H)$ is the corresponding Radon-Nikodým derivative process, we show that $\mathcal{L}_s^*(H)$ is a martingale; $\mathcal{L}_T^*(H)$ is almost surely positive; and $\mathbb{E}[\mathcal{L}_T^*(H)] = 1$ (see, e.g., Shreve, 2004, Theorem 1.6.1 and Definition 1.6.3).

Consider the following three cases to show that $\mathcal{L}_s^*(H)$ is a martingale (see, e.g., Borodin and Salminen, 2002, Chapter I), or $\mathbb{E}[\mathcal{L}_{s_2}^*(H)|\mathcal{F}_{s_1}] = \mathcal{L}_{s_1}^*(H)$, for all $0 \leq s_1 \leq s_2 \leq T$. First, for all $0 \leq s_1 \leq s_2 \leq H \leq T$, $\mathcal{L}_{s_1}^*(H) = \mathcal{L}_{s_2}^*(H) = 1$, hence $\mathbb{E}[\mathcal{L}_{s_2}^*(H)|\mathcal{F}_{s_1}] = \mathcal{L}_{s_1}^*(H)$. Second, for all $0 \leq H \leq s_1 \leq s_2 \leq T$,

$$\begin{aligned} \mathbb{E}[\mathcal{L}_{s_2}^*(H)|\mathcal{F}_{s_1}] &= \mathbb{E}\left[\frac{L_{s_2}^*}{L_H^*} \mid \mathcal{F}_{s_1}\right] = \mathbb{E}\left[\frac{\mathbb{E}[L_T^*|\mathcal{F}_{s_2}]}{L_H^*} \mid \mathcal{F}_{s_1}\right] = \frac{\mathbb{E}[\mathbb{E}[L_T^*|\mathcal{F}_{s_2}]|\mathcal{F}_{s_1}]}{L_H^*} \\ &= \frac{\mathbb{E}[L_T^*|\mathcal{F}_{s_1}]}{L_H^*} = \frac{L_{s_1}^*}{L_H^*} = \mathcal{L}_{s_1}^*(H). \end{aligned}$$

Third, for all $0 \leq s_1 \leq H \leq s_2 \leq T$,

$$\begin{aligned} \mathbb{E}[\mathcal{L}_{s_2}^*(H)|\mathcal{F}_{s_1}] &= \mathbb{E}\left[\frac{L_{s_2}^*}{L_H^*} \mid \mathcal{F}_{s_1}\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{L_{s_2}^*}{L_H^*} \mid \mathcal{F}_H\right] \mid \mathcal{F}_{s_1}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\frac{\mathbb{E}[L_T^*|\mathcal{F}_{s_2}]}{L_H^*} \mid \mathcal{F}_H\right] \mid \mathcal{F}_{s_1}\right] = \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{E}[L_T^*|\mathcal{F}_{s_2}]|\mathcal{F}_H]}{L_H^*} \mid \mathcal{F}_{s_1}\right] \\ &= \mathbb{E}\left[\frac{\mathbb{E}[L_T^*|\mathcal{F}_H]}{L_H^*} \mid \mathcal{F}_{s_1}\right] = \mathbb{E}\left[\frac{L_H^*}{L_H^*} \mid \mathcal{F}_{s_1}\right] = \mathcal{L}_{s_1}^*(H). \end{aligned}$$

Therefore, $\mathcal{L}_s^*(H)$ is a martingale.

Next, we show that $\mathcal{L}_T^*(H)$ is almost surely positive and $\mathbb{E}[\mathcal{L}_T^*(H)] = 1$. First, note that L_T^* is almost surely positive since \mathbb{P} and \mathbb{Q}^* are equivalent probability measures. Since $L_H^* = \mathbb{E}[L_T^*|\mathcal{F}_H]$, then L_H^* is also almost surely positive. Finally, by definition in equation (8), $\mathcal{L}_T^*(H) = L_T^*/L_H^*$ is also almost surely positive. Additionally, $\mathbb{E}[\mathcal{L}_T^*(H)] = \mathbb{E}[L_T^*/L_H^*] = \mathbb{E}[\mathbb{E}[L_T^*|\mathcal{F}_H]/L_H^*] = \mathbb{E}[L_H^*/L_H^*] = 1$. \square

B. Proof of Proposition 1

Proof. For the first property, for all $0 \leq t \leq H$, given $A \in \mathcal{F}_H$, we have $\mathbb{R}^*(A|\mathcal{F}_t) = \mathbb{E}_t^{\mathbb{R}^*} [\mathbf{1}_A] = \mathbb{E}_t^{\mathbb{P}} [\mathcal{L}_H^*(H)/\mathcal{L}_t^*(H) \cdot \mathbf{1}_A] = \mathbb{E}_t^{\mathbb{P}} [(L_H^*/L_H^*)/1 \cdot \mathbf{1}_A] = \mathbb{E}_t^{\mathbb{P}} [\mathbf{1}_A] = \mathbb{P}(A|\mathcal{F}_t)$.

For the second property, for all $H \leq s \leq T$, given $A \in \mathcal{F}_T$, we have

$$\begin{aligned} \mathbb{R}^*(A|\mathcal{F}_s) &= \mathbb{E}_s^{\mathbb{R}^*} [\mathbf{1}_A] = \mathbb{E}_s^{\mathbb{P}} \left[\frac{\mathcal{L}_T^*(H)}{\mathcal{L}_s^*(H)} \mathbf{1}_A \right] = \mathbb{E}_s^{\mathbb{P}} \left[\frac{L_T^*/L_H^*}{L_s^*/L_H^*} \mathbf{1}_A \right] = \mathbb{E}_s^{\mathbb{P}} \left[\frac{L_T^*}{L_s^*} \mathbf{1}_A \right] \\ &= \mathbb{E}_s^{\mathbb{Q}^*} [\mathbf{1}_A] = \mathbb{Q}^*(A|\mathcal{F}_s). \end{aligned}$$

The third property is a special case of the first property when $H = T$, while the fourth property is a special case of the second property when $H = t$.

For the last property, we have

$$\mathbb{E}_t^{\mathbb{R}^*} [Z_T] = \mathbb{E}_t^{\mathbb{R}^*} [\mathbb{E}_H^{\mathbb{R}^*} [Z_T]] = \mathbb{E}_t^{\mathbb{R}^*} [\mathbb{E}_H^{\mathbb{Q}^*} [Z_T]] = \mathbb{E}_t^{\mathbb{P}} [\mathbb{E}_H^{\mathbb{Q}^*} [Z_T]],$$

where the second equality is an application of the second property, the last equality is an application of the first property, since $\mathbb{E}_H^{\mathbb{Q}^*} [Z_T]$ is a random variable that is \mathcal{F}_H measurable. \square

C. Proof of Proposition 2

Proof. For the first property, for all $0 \leq H \leq s_1 \leq s_2 \leq T$,

$$\begin{aligned} \mathbb{E}_{s_1}^{\mathbb{R}^*} \left[\frac{F_{s_2}}{G_{s_2}} \right] &= \mathbb{E}_{s_1}^{\mathbb{P}} \left[\frac{\mathcal{L}_{s_2}^*(H)}{\mathcal{L}_{s_1}^*(H)} \cdot \frac{F_{s_2}}{G_{s_2}} \right] = \mathbb{E}_{s_1}^{\mathbb{P}} \left[\frac{L_{s_2}^*/L_H^*}{L_{s_1}^*/L_H^*} \cdot \frac{F_{s_2}}{G_{s_2}} \right] = \mathbb{E}_{s_1}^{\mathbb{P}} \left[\frac{L_{s_2}^*}{L_{s_1}^*} \cdot \frac{F_{s_2}}{G_{s_2}} \right] \\ &= \mathbb{E}_{s_1}^{\mathbb{Q}^*} \left[\frac{F_{s_2}}{G_{s_2}} \right] = \frac{F_{s_1}}{G_{s_1}}. \end{aligned}$$

For the second property, for all $0 \leq s_1 \leq s_2 \leq H \leq T$,

$$\mathbb{E}_{s_1}^{\mathbb{R}^*} \left[\frac{F_{s_2}}{G_{s_2}} \right] = \mathbb{E}_{s_1}^{\mathbb{P}} \left[\frac{\mathcal{L}_{s_2}^*(H)}{\mathcal{L}_{s_1}^*(H)} \cdot \frac{F_{s_2}}{G_{s_2}} \right] = \mathbb{E}_{s_1}^{\mathbb{P}} \left[\frac{1}{1} \cdot \frac{F_{s_2}}{G_{s_2}} \right] = \mathbb{E}_{s_1}^{\mathbb{P}} \left[\frac{F_{s_2}}{G_{s_2}} \right],$$

which is generally not equal to $\mathbb{E}_{s_1}^{\mathbb{Q}^*} [F_{s_2}/G_{s_2}]$, except when $\mathbb{P} = \mathbb{Q}^*$. \square

D. Proof of Lemma 2

Proof. We show that \mathbb{R}_1^* and \mathbb{P} are equivalent probability measures, and $\mathcal{L}_{1s}^*(H)$ is the corresponding Radon-Nikodým derivative process similar to the above process. First, we show that $\mathcal{L}_{1s}^*(H)$ is a martingale using the following three cases. For $0 \leq s_1 \leq s_2 \leq H \leq T$,

$$\mathbb{E} [\mathcal{L}_{1s_2}^*(H) | \mathcal{F}_{s_1}] = \mathbb{E} \left[\frac{\mathbb{E}_{s_2} [G_H]}{\mathbb{E}_0 [G_H]} \mid \mathcal{F}_{s_1} \right] = \frac{\mathbb{E}_{s_1} [\mathbb{E}_{s_2} [G_H]]}{\mathbb{E}_0 [G_H]} = \frac{\mathbb{E}_{s_1} [G_H]}{\mathbb{E}_0 [G_H]} = \mathcal{L}_{1s_1}^*(H).$$

For $0 \leq H \leq s_1 \leq s_2 \leq T$,

$$\begin{aligned} \mathbb{E} [\mathcal{L}_{1s_2}^*(H) | \mathcal{F}_{s_1}] &= \mathbb{E} \left[\frac{G_H}{\mathbb{E}_0 [G_H]} \cdot \frac{L_{s_2}^*}{L_H^*} \mid \mathcal{F}_{s_1} \right] = \frac{G_H}{\mathbb{E}_0 [G_H]} \cdot \frac{\mathbb{E}_{s_1} [L_{s_2}^*]}{L_H^*} = \frac{G_H}{\mathbb{E}_0 [G_H]} \cdot \frac{L_{s_1}^*}{L_H^*} \\ &= \mathcal{L}_{1s_1}^*(H). \end{aligned}$$

For $0 \leq s_1 \leq H \leq s_2 \leq T$,

$$\begin{aligned} \mathbb{E} [\mathcal{L}_{1s_2}^*(H) | \mathcal{F}_{s_1}] &= \mathbb{E} \left[\frac{G_H}{\mathbb{E}_0 [G_H]} \cdot \frac{L_{s_2}^*}{L_H^*} \mid \mathcal{F}_{s_1} \right] = \mathbb{E}_{s_1} \left[\mathbb{E}_H \left[\frac{G_H}{\mathbb{E}_0 [G_H]} \cdot \frac{L_{s_2}^*}{L_H^*} \right] \right] \\ &= \mathbb{E}_{s_1} \left[\frac{G_H}{\mathbb{E}_0 [G_H]} \cdot \frac{\mathbb{E}_H [L_{s_2}^*]}{L_H^*} \right] = \mathbb{E}_{s_1} \left[\frac{G_H}{\mathbb{E}_0 [G_H]} \cdot \frac{L_H^*}{L_H^*} \right] = \frac{\mathbb{E}_{s_1} [G_H]}{\mathbb{E}_0 [G_H]} \\ &= \mathcal{L}_{1s_1}^*(H). \end{aligned}$$

Next, we show that $\mathcal{L}_{1T}^*(H)$ is almost surely positive and $\mathbb{E}[\mathcal{L}_{1T}^*(H)] = 1$. Since the numeraire asset G is positive, using similar logic as in the proof of Lemma 1, $\mathcal{L}_{1T}^*(H)$ is almost surely positive, and

$$\begin{aligned} \mathbb{E} [\mathcal{L}_{1T}^*(H)] &= \mathbb{E}_0 \left[\frac{G_H}{\mathbb{E}_0 [G_H]} \cdot \frac{L_T^*}{L_H^*} \right] = \mathbb{E}_0 \left[\mathbb{E}_H \left[\frac{G_H}{\mathbb{E}_0 [G_H]} \cdot \frac{L_T^*}{L_H^*} \right] \right] \\ &= \mathbb{E}_0 \left[\frac{G_H}{\mathbb{E}_0 [G_H]} \cdot \frac{\mathbb{E}_H [L_T^*]}{L_H^*} \right] = \mathbb{E}_0 \left[\frac{G_H}{\mathbb{E}_0 [G_H]} \cdot \frac{L_H^*}{L_H^*} \right] = 1. \end{aligned}$$

□

E. Proof of Theorem 3

Proof. Using the future state price density $f_H(S_T|Y_H)$ at time H , where Y_H represents the state variable vector at time H , the future price of a European call option expiring at time T , can be obtained as

$$C_H = \int_K^\infty (S - K) f_H(S|Y_H) dS. \quad (\text{A1})$$

Let $p_t^{\mathbb{P}}(Y_H)$ be the probability density of the state variable vector Y_H conditioned on time t information under \mathbb{P} . For any \mathcal{F}_H -measurable variable $Z_H = Z_H(Y_H)$, which depends on the state variable vector Y_H , its expectation under \mathbb{P} is

$$\mathbb{E}_t[Z_H] = \int Z_H p_t^{\mathbb{P}}(Y_H) dY_H. \quad (\text{A2})$$

The expected future price of the call option can be simplified as

$$\begin{aligned} \mathbb{E}_t[C_H] &= \int C_H p_t^{\mathbb{P}}(Y_H) dY_H, \\ &= \int \left(\int_K^\infty (S - K) f_H(S|Y_H) dS \right) p_t^{\mathbb{P}}(Y_H) dY_H \\ &= \int_K^\infty (S - K) \left(\int f_H(S|Y_H) p_t^{\mathbb{P}}(Y_H) dY_H \right) dS \\ &= \int_K^\infty (S - K) \mathbb{E}_t[f_H(S|Y_H)] dS, \end{aligned} \quad (\text{A3})$$

where the first equality uses equation (A2), the second equality uses equation (A1), the third equality changes the integral order, and the last equality uses equation (A2) again.

Taking the second derivative of $\mathbb{E}_t(C_H)$ with respect to K gives

$$\mathbb{E}_t[f_H(K|Y_H)] = \frac{\partial^2 \mathbb{E}_t[C_H]}{\partial K^2} \quad \text{or,} \quad \mathbb{E}_t[f_H(S_T|Y_H)] = \left[\frac{\partial^2 \mathbb{E}_t[C_H]}{\partial K^2} \right]_{K=S_T},$$

which proves the first part of equation (43) in Theorem 3. The same logic applies to the case of the put option, so we omit the proof.

Recalling equation (34) in Corollary 2, we can also write the expected future price of the call option as

$$\begin{aligned}\mathbb{E}_t[C_H] &= \mathbb{E}_t^{\mathbb{P}} [P(H, T)] \mathbb{E}_t^{\mathbb{R}_1^T} [(S_T - K)^+] \\ &= \mathbb{E}_t^{\mathbb{P}} [P(H, T)] \int_K^\infty (S - K) p_t^{\mathbb{R}_1^T}(S) dS,\end{aligned}\tag{A4}$$

where $p_t^{\mathbb{R}_1^T}(S_T)$ is time t probability density of S_T under \mathbb{R}_1^T .

Due to the equivalence of equations (A3) and (A4), we have

$$g_t(H, S_T) \triangleq \mathbb{E}_t [f_H(S_T|Y_H)] = \mathbb{E}_t^{\mathbb{P}} [P(H, T)] p_t^{\mathbb{R}_1^T}(S_T),$$

which proves the second part of equation (43) in Theorem 3. Equation (44) is a special case of equation (43), and we omit the proof. \square

Appendix B. One-dimensional Brownian Motion under the \mathbb{R} Measure

Consider the Radon-Nikodým derivative processes $\mathcal{L}_s^*(H)$ and L_s^* in Lemma 1 and denote them as $\mathcal{L}_s(H)$ and L_s , respectively, for the special case when the numeraire is the money market account, or $G = B \triangleq e^{\int_0^s r_u du}$. For this special case, $\mathbb{R}^* = \mathbb{R}$, as defined in Corollary 1, and we know that L_s under the Black and Scholes (1973) model is given as

$$L_s \triangleq \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_s} = \exp \left(- \int_0^s \gamma dW_u^{\mathbb{P}} - \frac{1}{2} \int_0^s \gamma^2 du \right),\tag{B1}$$

where $\gamma = (\mu - r) / \sigma$ is the market price of risk (MPR) and r is the constant risk-free rate.

Using equations (8) and (B1), $\mathcal{L}_s(H)$ can be given as follows:

$$\begin{aligned}\mathcal{L}_s(H) &\triangleq \frac{d\mathbb{R}}{d\mathbb{P}} \Big|_{\mathcal{F}_s} = \begin{cases} \frac{L_s}{L_H} = \exp\left(-\int_H^s \gamma dW_u^{\mathbb{P}} - \frac{1}{2} \int_H^s \gamma^2 du\right), & \text{if } H \leq s \leq T, \\ 1, & \text{if } 0 \leq s < H, \end{cases} \\ &= \exp\left(-\int_0^s \gamma_u(H) dW_u^{\mathbb{P}} - \frac{1}{2} \int_0^s (\gamma_u(H))^2 du\right),\end{aligned}$$

where

$$\gamma_s(H) \triangleq \mathbb{1}_{\{s \geq H\}} \gamma = \begin{cases} \gamma, & \text{if } H \leq s \leq T, \\ 0, & \text{if } 0 \leq s < H. \end{cases}$$

The Girsanov theorem allows the transformation of Brownian motions under two equivalent probability measures. Therefore, given \mathbb{Q} and \mathbb{P} , we know that $W_s^{\mathbb{Q}} = W_s^{\mathbb{P}} + \int_0^s \gamma du$. Similarly, the relationship of the Brownian motions under the \mathbb{R} and \mathbb{P} measures is given as

$$W_s^{\mathbb{R}} = W_s^{\mathbb{P}} + \int_0^s \gamma_u(H) du = W_s^{\mathbb{P}} + \int_0^s \mathbb{1}_{\{u \geq H\}} \gamma du.$$

Appendix C. Expected Future Option Price under the Black-Scholes Model

The expected European call price under the Black-Scholes model is given as

$$\mathbb{E}_t[C_H] = S_t e^{\mu(H-t)} \mathcal{N}(\hat{d}_1) - K e^{-r(T-H)} \mathcal{N}(\hat{d}_2), \quad (\text{C1})$$

where $\mathcal{N}(\cdot)$ is the standard normal cumulative distribution function, and

$$\begin{aligned}\hat{d}_1 &= \frac{\ln(S_t/K) + \mu(H-t) + r(T-H) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \\ \hat{d}_2 &= \frac{\ln(S_t/K) + \mu(H-t) + r(T-H) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.\end{aligned}$$

Appendix D. Expected Future Price of Forward-Start Option using the Black-Scholes Model

Consider a forward-start call option (FSO) with a strike price kS_{T_0} determined at a future time T_0 (for any $T_0 \leq T$), and a terminal payoff $(S_T - kS_{T_0})$ at time T . Let $C(S_t, K, \sigma, r, t, T) \triangleq C_t = \mathbb{E}_t^{\mathbb{Q}} [e^{-r(T-t)}(S_T - K)^+]$ be the Black-Scholes call price function. Using the homogeneity property of the call price, Rubinstein (1991) obtains the price of the FSO as follows:

$$\begin{aligned} \text{FSO}_t &= \mathbb{E}_t^{\mathbb{Q}} [e^{-r(T_0-t)} \mathbb{E}_{T_0}^{\mathbb{Q}} [e^{-r(T-T_0)}(S_T - kS_{T_0})^+]] \\ &= \mathbb{E}_t^{\mathbb{Q}} [e^{-r(T_0-t)} C(S_{T_0}, kS_{T_0}, \sigma, r, T_0, T)] \\ &= \mathbb{E}_t^{\mathbb{Q}} [e^{-r(T_0-t)} S_{T_0} C(1, k, \sigma, r, T_0, T)] \\ &= S_t C(1, k, \sigma, r, T_0, T). \end{aligned}$$

Since the FSO can be interpreted as a constant share $C(1, k, \sigma, r, T_0, T)$ of the asset S , until time T_0 , the expected future price of the FSO when $t \leq H \leq T_0 \leq T$, obtains trivially by taking the expected future value of the asset at time H , as follows:³²

$$\mathbb{E}_t[\text{FSO}_H] = S_t e^{\mu(H-t)} C(1, k, \sigma, r, T_0, T). \quad (\text{D1})$$

The non-trivial case occurs when $t \leq T_0 \leq H \leq T$. Under this case, the \mathbb{R} measure can be used to compute the expected future price of the FSO by applying Corollary 1 as follows:

$$\mathbb{E}_t[\text{FSO}_H] = \mathbb{E}_t^{\mathbb{R}} [e^{-r(T-H)}(S_T - kS_{T_0})^+].$$

Let $\bar{C}(S_t, K, \sigma, r, \mu, t, H, T) \triangleq \mathbb{E}_t[C_H] = \mathbb{E}_t^{\mathbb{R}} [e^{-r(T-H)}(S_T - K)^+]$ be the Black-Scholes expected future call price function given by equation (C1). Using the

³²Alternatively, the result in equation (D1) can also be derived by applying the \mathbb{R} measure using Corollary 1.

homogeneity property of the expected future call price, and using Proposition 1(i), we can solve the expected future price of the FSO given in the above equation as follows:

$$\begin{aligned}
\mathbb{E}_t[\text{FSO}_H] &= \mathbb{E}_t^{\mathbb{R}} \left[\mathbb{E}_{T_0}^{\mathbb{R}} \left[e^{-r(T-H)} (S_T - kS_{T_0})^+ \right] \right] \\
&= \mathbb{E}_t^{\mathbb{R}} \left[\bar{C}(S_{T_0}, kS_{T_0}, \sigma, r, \mu, T_0, H, T) \right] \\
&= \mathbb{E}_t^{\mathbb{R}} \left[S_{T_0} \bar{C}(1, k, \sigma, r, \mu, T_0, H, T) \right] \\
&= \mathbb{E}_t^{\mathbb{P}} \left[S_{T_0} \bar{C}(1, k, \sigma, r, \mu, T_0, H, T) \right] \\
&= S_t e^{\mu(T_0-t)} \bar{C}(1, k, \sigma, r, \mu, T_0, H, T). \tag{D2}
\end{aligned}$$

Together, equations (D1) and (D2) provide the full solution to the expected future price of the forward-start call option using: i) the Black-Scholes call price function when $H \leq T_0$, and ii) the Black-Scholes expected future call price function when $H \geq T_0$, respectively.

Appendix E. Solutions of the R -transform and the Extended R -transform

Here we provide the solutions of the R -transform and the extended R -transform by giving the following two propositions:

PROPOSITION A1: *Under technical regularity conditions given in Duffie et al. (2000), the R -transform ϕ^R of Y defined by equation (32) is given by*

$$\phi^R(z; t, T, H) = \exp \left(-A_{\chi}^{(\mathbf{b}, \mathbf{c})}(H-t) - B_{\chi}^{(\mathbf{b}, \mathbf{c})}(H-t)' Y_t \right), \tag{E1}$$

where $\mathbf{b} = \left(A_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(T-H), B_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(T-H) \right)$, $\mathbf{c} = (0, 0)$, $\mathbf{b}^* = (0, -z)$, and $\mathbf{c}^* = (\rho_0, \rho_1)$.

For any well-defined $\mathbf{b} \triangleq (b_0, b_1) \in \mathcal{C}^{N+1}$, $\mathbf{c} \triangleq (c_0, c_1) \in \mathcal{R}^{N+1}$ and a coefficient characteristic $\chi = (k, h, l, \theta)$, $A_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)$ and $B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)$ are defined by the solution of

the following expectation with $\tau < H - t$,

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(-b_0 - b_1' Y_{t+\tau} - \int_t^{t+\tau} (c_0 + c_1' Y_u) du \right) \right] \\ &= \mathbb{E}_t^{\mathbb{P}} \left[\exp \left(-b_0 - b_1' Y_{t+\tau} - \int_t^{t+\tau} (c_0 + c_1' Y_u) du \right) \right] \\ &= \exp \left(-A_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau) - B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)' Y_t \right), \end{aligned}$$

and are obtained by solving the following complex-valued ODEs:

$$\begin{aligned} \frac{dA_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)}{d\tau} &= k_0' B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau) - \frac{1}{2} B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)' h_0 B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau) - l_0 (\theta (-B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)) - 1) + c_0, \\ \frac{dB_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)}{d\tau} &= k_1' B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau) - \frac{1}{2} B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)' h_1 B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau) - l_1 (\theta (-B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)) - 1) + c_1, \end{aligned} \tag{E2}$$

with boundary conditions $A_{\chi}^{(\mathbf{b}, \mathbf{c})}(0) = b_0$, and $B_{\chi}^{(\mathbf{b}, \mathbf{c})}(0) = b_1$. The solutions can be found numerically, for example by Runge-Kutta. $A_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(\tau)$ and $B_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(\tau)$ are defined by the solution of the following expectation,

$$\begin{aligned} & \mathbb{E}_H^{\mathbb{R}} \left[\exp \left(-b_0 - b_1' Y_{H+\tau} - \int_H^{H+\tau} (c_0 + c_1' Y_u) du \right) \right] \\ &= \mathbb{E}_H^{\mathbb{Q}} \left[\exp \left(-b_0 - b_1' Y_{H+\tau} - \int_H^{H+\tau} (c_0 + c_1' Y_u) du \right) \right] \\ &= \exp \left(-A_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(\tau) - B_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(\tau)' Y_H \right), \end{aligned}$$

and are obtained similarly as in equation (E2) with the coefficient characteristic $\chi = (k, h, l, \theta)$ replaced with a coefficient characteristic $\chi^* = (k^*, h^*, l^*, \theta^*)$, and with the parameter \mathbf{b} and \mathbf{c} replaced with $\mathbf{b}^* \triangleq (b_0^*, b_1^*)$ and $\mathbf{c}^* \triangleq (c_0^*, c_1^*)$.

Proof. Under the integration conditions in the definition of Duffie et al. (2000, page 1351), we can prove that the solution of the expectation $\mathbb{E}_t^{\mathbb{P}} \left[\exp \left(-b_0 - b_1' Y_{t+\tau} - \int_t^{t+\tau} (c_0 + c_1' Y_u) du \right) \right]$ is $\exp \left(-A_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau) - B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)' Y_t \right)$, and the solution of the expectation $\mathbb{E}_H^{\mathbb{Q}} \left[\exp \left(-b_0 - b_1' Y_{H+\tau} - \int_H^{H+\tau} (c_0 + c_1' Y_u) du \right) \right]$ is $\exp \left(-A_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(\tau) - B_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(\tau)' Y_H \right)$, by following

the proof of Proposition 1 in Duffie et al. (2000). Note that under the \mathbb{R} measure, the coefficient characteristic is χ from t to H and χ^* from H to T . Therefore, for the R -transform

$$\begin{aligned}\phi^R(z; t, T, H) &\triangleq \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T r(Y_u) du \right) \exp(z' Y_T) \right] \\ &= \mathbb{E}_t^{\mathbb{R}} \left[\mathbb{E}_H^{\mathbb{R}} \left[\exp \left(z' Y_T - \int_H^T (\rho_0 + \rho_1' Y_u) du \right) \right] \right] \\ &= \mathbb{E}_t^{\mathbb{P}} \left[\mathbb{E}_H^{\mathbb{Q}} \left[\exp \left(z' Y_T - \int_H^T (\rho_0 + \rho_1' Y_u) du \right) \right] \right] \\ &= \mathbb{E}_t^{\mathbb{P}} \left[\exp \left(-A_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(T-H) - B_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(T-H)' Y_H \right) \right] \\ &= \exp \left(-A_{\chi}^{(\mathbf{b}, \mathbf{c})}(H-t) - B_{\chi}^{(\mathbf{b}, \mathbf{c})}(H-t)' Y_t \right),\end{aligned}$$

with $\mathbf{b}^* = (0, -z)$, $\mathbf{c}^* = (\rho_0, \rho_1)$, $\mathbf{b} = \left(A_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(T-H), B_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(T-H) \right)$, and $\mathbf{c} = (0, 0)$. \square

PROPOSITION A2: Under technical regularity conditions given in Duffie et al. (2000), the extended R -transform φ^R of Y defined by equation (33) is given by

$$\varphi^R(v, z; t, T, H) = \phi^R(z; t, T, H) \left(D_{\chi}^{(\mathbf{b}, \mathbf{c}, \mathbf{d})}(H-t) + E_{\chi}^{(\mathbf{b}, \mathbf{c}, \mathbf{d})}(H-t)' Y_t \right), \quad (\text{E3})$$

where $\mathbf{b} = \left(A_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(T-H), B_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*)}(T-H) \right)$, $\mathbf{c} = (0, 0)$, $\mathbf{d} = \left(D_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*, \mathbf{d}^*)}(T-H), E_{\chi^*}^{(\mathbf{b}^*, \mathbf{c}^*, \mathbf{d}^*)}(T-H) \right)$, $\mathbf{b}^* = (0, -z)$, $\mathbf{c}^* = (\rho_0, \rho_1)$, and $\mathbf{d}^* = (0, v)$.

For any well-defined $\mathbf{b} \triangleq (b_0, b_1) \in \mathcal{C}^{N+1}$, $\mathbf{c} \triangleq (c_0, c_1) \in \mathcal{R}^{N+1}$, $\mathbf{d} \triangleq (d_0, d_1) \in \mathcal{R}^{N+1}$, and a coefficient characteristic $\chi = (k, h, l, \theta)$, $A_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)$, $B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)$, $D_{\chi}^{(\mathbf{b}, \mathbf{c}, \mathbf{d})}(\tau)$ and $E_{\chi}^{(\mathbf{b}, \mathbf{c}, \mathbf{d})}(\tau)$ are defined by the solution of the following expectation with $\tau < H - t$,

$$\begin{aligned}&\mathbb{E}_t^{\mathbb{R}} \left[\exp \left(-b_0 - b_1' Y_{t+\tau} - \int_t^{t+\tau} (c_0 + c_1' Y_u) du \right) (d_0 + d_1' Y_{t+\tau}) \right] \\ &= \mathbb{E}_t^{\mathbb{P}} \left[\exp \left(-b_0 - b_1' Y_{t+\tau} - \int_t^{t+\tau} (c_0 + c_1' Y_u) du \right) (d_0 + d_1' Y_{t+\tau}) \right] \\ &= \exp \left(-A_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau) - B_{\chi}^{(\mathbf{b}, \mathbf{c})}(\tau)' Y_t \right) \cdot \left(D_{\chi}^{(\mathbf{b}, \mathbf{c}, \mathbf{d})}(\tau) + E_{\chi}^{(\mathbf{b}, \mathbf{c}, \mathbf{d})}(\tau)' Y_t \right),\end{aligned}$$

where $A_\chi^{(\mathbf{b},\mathbf{c})}(\tau)$ and $B_\chi^{(\mathbf{b},\mathbf{c})}(\tau)$ are obtained as in Proposition A1, and $D_\chi^{(\mathbf{b},\mathbf{c},\mathbf{d})}(\tau)$ and $E_\chi^{(\mathbf{b},\mathbf{c},\mathbf{d})}(\tau)$ satisfy the complex-valued ODEs

$$\begin{aligned}\frac{dD_\chi^{(\mathbf{b},\mathbf{c},\mathbf{d})}(\tau)}{d\tau} &= k'_0 E_\chi^{(\mathbf{b},\mathbf{c},\mathbf{d})}(\tau) - B_\chi^{(\mathbf{b},\mathbf{c})}(\tau)' h_0 E_\chi^{(\mathbf{b},\mathbf{c},\mathbf{d})}(\tau) + l_0 \nabla\theta(-B_\chi^{(\mathbf{b},\mathbf{c})}(\tau)) E_\chi^{(\mathbf{b},\mathbf{c},\mathbf{d})}(\tau), \\ \frac{dE_\chi^{(\mathbf{b},\mathbf{c},\mathbf{d})}(\tau)}{d\tau} &= k'_1 E_\chi^{(\mathbf{b},\mathbf{c},\mathbf{d})}(\tau) - B_\chi^{(\mathbf{b},\mathbf{c})}(\tau)' h_1 E_\chi^{(\mathbf{b},\mathbf{c},\mathbf{d})}(\tau) + l_1 \nabla\theta(-B_\chi^{(\mathbf{b},\mathbf{c})}(\tau)) E_\chi^{(\mathbf{b},\mathbf{c},\mathbf{d})}(\tau),\end{aligned}\tag{E4}$$

with boundary conditions $D_\chi^{(\mathbf{b},\mathbf{c},\mathbf{d})}(0) = d_0$, and $E_\chi^{(\mathbf{b},\mathbf{c},\mathbf{d})}(0) = d_1$, and $\nabla\theta(x)$ is gradient of $\theta(x)$ with respect to $x \in \mathcal{C}^N$. $A_{\chi^*}^{(\mathbf{b}^*,\mathbf{c}^*)}(\tau)$, $B_{\chi^*}^{(\mathbf{b}^*,\mathbf{c}^*)}(\tau)$, $D_{\chi^*}^{(\mathbf{b}^*,\mathbf{c}^*,\mathbf{d}^*)}(\tau)$ and $E_{\chi^*}^{(\mathbf{b}^*,\mathbf{c}^*,\mathbf{d}^*)}(\tau)$ are defined by the solution of the following expectation,

$$\begin{aligned}& \mathbb{E}_H^{\mathbb{R}} \left[\exp \left(-b_0 - b_1' Y_{H+\tau} - \int_H^{H+\tau} (c_0 + c_1' Y_u) du \right) (d_0 + d_1' Y_{H+\tau}) \right] \\ &= \mathbb{E}_H^{\mathbb{Q}} \left[\exp \left(-b_0 - b_1' Y_{H+\tau} - \int_H^{H+\tau} (c_0 + c_1' Y_u) du \right) (d_0 + d_1' Y_{H+\tau}) \right] \\ &= \exp \left(-A_{\chi^*}^{(\mathbf{b}^*,\mathbf{c}^*)}(\tau) - B_{\chi^*}^{(\mathbf{b}^*,\mathbf{c}^*)}(\tau)' Y_H \right) \cdot \left(D_{\chi^*}^{(\mathbf{b}^*,\mathbf{c}^*,\mathbf{d}^*)}(\tau) + E_{\chi^*}^{(\mathbf{b}^*,\mathbf{c}^*,\mathbf{d}^*)}(\tau)' Y_H \right),\end{aligned}$$

where $A_{\chi^*}^{(\mathbf{b}^*,\mathbf{c}^*)}(\tau)$ and $B_{\chi^*}^{(\mathbf{b}^*,\mathbf{c}^*)}(\tau)$ are obtained as in Proposition A1, and $D_{\chi^*}^{(\mathbf{b}^*,\mathbf{c}^*,\mathbf{d}^*)}(\tau)$ and $E_{\chi^*}^{(\mathbf{b}^*,\mathbf{c}^*,\mathbf{d}^*)}(\tau)$ are obtained similarly as in equation (E4) with the coefficient characteristic $\chi = (k, h, l, \theta)$ replaced with a coefficient characteristic $\chi^* = (k^*, h^*, l^*, \theta^*)$, and with the parameter \mathbf{b} , \mathbf{c} , and \mathbf{d} replaced with $\mathbf{b}^* \triangleq (b_0^*, b_1^*)$, $\mathbf{c}^* \triangleq (c_0^*, c_1^*)$, and $\mathbf{d}^* \triangleq (d_0^*, d_1^*)$.

Proof. The proof follows similarly from the proof of Proposition A1, we omit the specific steps. \square

Appendix F. Expected Future Corporate Bond Price under the CDG Model

To compute the default probability term $\mathbb{E}_t^{\mathbb{R}^T} \left[\mathbf{1}_{\{\tau_{[t,T]} < T\}} \right]$ in equation (42), we follow the method of Mueller (2000) which requires the conditional mean and conditional variance of the log-leverage ratio l . As can be seen from equation (41),

the variable l follows a Gaussian distribution under the \mathbb{R}_1^T measure. Therefore, we have

$$\mathbb{E}_t^{\mathbb{R}_1^T} \left[\mathbb{1}_{\{\tau_{[t,T]} < T\}} \right] = \Pr(\tau_{[t,T]} < T) = \sum_{i=1}^n q_i (\tau_{\ln K} = t + t_i | l_t, r_t, t) = \sum_{i=1}^n q_i,$$

where

$$q_1 = \frac{\mathcal{N}(a_1)}{\mathcal{N}(b_{1,1})}, \quad q_i = \frac{\mathcal{N}(a_i) - \sum_{j=1}^{i-1} \mathcal{N}(b_{i,j}) \times q_j}{\mathcal{N}(b_{i,i})},$$

with

$$\begin{aligned} \mathcal{N}(a_i) &\triangleq \Pr(l_{t+i\Delta} \geq \ln K | l_t, r_t, t) \\ &= \mathcal{N}\left(\frac{M(t, t+i\Delta, H, T) - \ln K}{\sqrt{S(t, t+i\Delta)}}\right), \\ \mathcal{N}(b_{i,j}) &\triangleq \Pr(l_{t+i\Delta} \geq \ln K | l_{t+t_j} = \ln K, l_t, r_t, t) \\ &= \mathcal{N}\left(\frac{\tilde{M}(t, t+i\Delta, t+t_j, \ln K, H, T) - \ln K}{\sqrt{\tilde{S}(t, t+i\Delta, t_j)}}\right). \end{aligned}$$

The log-leverage ratio $(l_u | l_t, r_t, t) \sim \mathcal{N}(M(t, u, H, T), S(t, u))$, for $t < u < T$. The conditional mean $M(t, u, H, T)$ and the conditional variance $S(t, u)$ can be computed as

$$\begin{aligned} M(t, u, H, T) &= l_t e^{-\lambda(u-t)} - (1 + \lambda\phi) (r_t - m_r) e^{-\alpha_r(u-t)} B_{(\lambda-\alpha_r)}(u-t) \\ &\quad - \left(\frac{\rho\sigma\sigma_r}{\alpha_r} + (1 + \lambda\phi) \frac{\sigma_r^2}{2\alpha_r^2} \right) (e^{-\alpha_r(T-u)} - e^{-\alpha_r(H-u)}) B_{(\lambda+\alpha_r)}(u-t) \\ &\quad + (1 + \lambda\phi) \frac{\sigma_r^2}{2\alpha_r^2} (e^{-\alpha_r(T-u)} - e^{-\alpha_r(H-u)}) e^{-2\alpha_r(u-t)} B_{(\lambda-\alpha_r)}(u-t) \\ &\quad + (\lambda\bar{l} - \sigma\gamma^S - (1 + \lambda\phi) m_r) B_\lambda(u-t) \\ &\quad + \mathbb{1}_{\{u \geq H\}} \left[- (1 + \lambda\phi) \left(\frac{\sigma_r^2}{2\alpha_r^2} + \frac{\sigma_r \gamma_r}{\alpha_r} \right) e^{-\alpha_r(u-H)} B_{(\lambda-\alpha_r)}(u-H) \right. \\ &\quad \left. - \left(\frac{\rho\sigma\sigma_r}{\alpha_r} + (1 + \lambda\phi) \frac{\sigma_r^2}{2\alpha_r^2} \right) e^{-\alpha_r(H-u)} B_{(\lambda+\alpha_r)}(u-H) \right] \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{\rho\sigma\sigma_r}{\alpha_r} + \sigma\gamma^S + (1 + \lambda\phi) \left(\frac{\sigma_r^2}{\alpha_r^2} + \frac{\sigma_r\gamma_r}{\alpha_r} \right) \right) B_\lambda(u - H) \Big], \\
S(t, u) & = \left(\frac{(1 + \lambda\phi)\sigma_r}{\lambda - \alpha_r} \right)^2 B_{2\alpha_r}(u - t) \\
& + \left(\sigma^2 + \left(\frac{(1 + \lambda\phi)\sigma_r}{\lambda - \alpha_r} \right)^2 - \frac{2\rho\sigma\sigma_r(1 + \lambda\phi)}{\lambda - \alpha_r} \right) B_{2\lambda}(u - t) \\
& + 2 \left(\frac{\rho\sigma\sigma_r(1 + \lambda\phi)}{\lambda - \alpha_r} - \left(\frac{(1 + \lambda\phi)\sigma_r}{\lambda - \alpha_r} \right)^2 \right) B_{(\lambda + \alpha_r)}(u - t).
\end{aligned}$$

In addition, $(l_u|l_s, l_t, r_t, t) \sim \mathcal{N} \left(\tilde{M}(t, u, s, l_s, H, T), \tilde{S}(t, u, s) \right)$, for $u > s$, which can be obtained as

$$\begin{aligned}
\tilde{M}(t, u, s, l_s, H, T) & = M(t, u, H, T) + \frac{V(t, u, s)}{S(t, s)} \times (l_s - M(t, s, H, T)), \\
\tilde{S}(t, u, s) & = S(t, u) \times \left(1 - \frac{V(t, u, s)^2}{S(t, u)S(t, s)} \right),
\end{aligned}$$

where $V(t, u, s)$ is the covariance of the random variables $(l_u|l_t, r_t, t)$ and $(l_s|l_t, r_t, t)$, given as

$$\begin{aligned}
V(t, u, s) & = \left(\frac{(1 + \lambda\phi)\sigma_r}{\lambda - \alpha_r} \right)^2 \frac{e^{-\alpha_r(u-s)} - e^{-\alpha_r(u+s-2t)}}{2\alpha_r} \\
& + \left(\sigma^2 + \left(\frac{(1 + \lambda\phi)\sigma_r}{\lambda - \alpha_r} \right)^2 - \frac{2\rho\sigma\sigma_r(1 + \lambda\phi)}{\lambda - \alpha_r} \right) \frac{e^{-\lambda(u-s)} - e^{-\lambda(u+s-2t)}}{2\lambda} \\
& + \left(\frac{\rho\sigma\sigma_r(1 + \lambda\phi)}{\lambda - \alpha_r} - \left(\frac{(1 + \lambda\phi)\sigma_r}{\lambda - \alpha_r} \right)^2 \right) \\
& \times \frac{e^{-\alpha_r(u-s)} - e^{-\alpha_r(u-t) - \lambda(s-t)} - e^{-\alpha_r(s-t) - \lambda(u-t)} + e^{-\lambda(u-s)}}{\lambda + \alpha_r}.
\end{aligned}$$

Furthermore, $\mathbb{E}_t^{\mathbb{P}} [P(H, T)]$ in equation (42) is the expected future bond price under the Vasicek model, which can be solved using the results derived in Section II.B, and the solution is given in Internet Appendix Section II.A (see equation IA4).

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Internet Appendix for “A Theory of Equivalent Expectation Measures for Contingent Claim Returns”

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ABSTRACT

This Internet Appendix provides the following supplemental sections to the main article:

- Section **I**. Additional Theoretical Results Related to Different EEMs
- Section **II**. Expected Future Bond Prices under Specific ATSMs
- Section **III**. Expected Future Bond Price under the Quadratic Term Structure Model
- Section **IV**. Expected Future Price of Forward-Start Option using the Forward-Start R -Transform
- Section **V**. Expected Future Option Price under the Merton Model with Vasicek Short Rate Process
- Section **VI**. Expected Future Price of an Asset Exchange Option
- Section **VII**. A Procedure to Extract the Expected FSPD

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I. Additional Theoretical Results Related to Different EEMs

A. Properties of the \mathbb{R}_1^* Measure

The properties of the \mathbb{R}_1^* measure is given by the following proposition:

PROPOSITION IA1: *For all $0 \leq t \leq H \leq s \leq T$, the \mathbb{R}_1^* measure has the following properties:*

- (i) $\mathbb{R}_1^*(A | \mathcal{F}_t) = \mathbb{E}_t^{\mathbb{P}} \left[\frac{G_H}{\mathbb{E}_t^{\mathbb{P}}[G_H]} \mathbf{1}_A \right]$ for all $A \in \mathcal{F}_H$.
- (ii) $\mathbb{R}_1^*(A | \mathcal{F}_s) = \mathbb{R}^*(A | \mathcal{F}_s) = \mathbb{Q}^*(A | \mathcal{F}_s)$ for all $A \in \mathcal{F}_T$.
- (iii) When $H = T$, $\mathbb{R}_1^*(A | \mathcal{F}_t) = \mathbb{E}_t^{\mathbb{P}} \left[\frac{G_T}{\mathbb{E}_t^{\mathbb{P}}[G_T]} \mathbf{1}_A \right]$ for all $A \in \mathcal{F}_T$.
- (iv) When $H = t$, $\mathbb{R}_1^*(A | \mathcal{F}_t) = \mathbb{R}^*(A | \mathcal{F}_t) = \mathbb{Q}^*(A | \mathcal{F}_t)$ for all $A \in \mathcal{F}_T$.
- (v) $\mathbb{E}_t^{\mathbb{R}_1^*} [Z_T] = \mathbb{E}_t^{\mathbb{R}^*} \left[\frac{G_H}{\mathbb{E}_t^{\mathbb{R}^*}[G_H]} \cdot Z_T \right] = \mathbb{E}_t^{\mathbb{P}} \left[\frac{G_H}{\mathbb{E}_t^{\mathbb{P}}[G_H]} \mathbb{E}_H^{\mathbb{Q}^*} [Z_T] \right]$ for any random variable Z_T at time T .

Proof. For the first property, for all $0 \leq t \leq H$, given $A \in \mathcal{F}_H$, we have

$$\begin{aligned} \mathbb{R}_1^*(A | \mathcal{F}_t) &= \mathbb{E}_t^{\mathbb{R}_1^*} [\mathbf{1}_A] = \mathbb{E}_t^{\mathbb{P}} \left[\frac{\mathcal{L}_{1H}^*(H)}{\mathcal{L}_{1t}^*(H)} \cdot \mathbf{1}_A \right] = \mathbb{E}_t^{\mathbb{P}} \left[\frac{G_H / \mathbb{E}_0^{\mathbb{P}}[G_H] \cdot L_H^* / L_H^*}{\mathbb{E}_t^{\mathbb{P}}[G_H] / \mathbb{E}_0^{\mathbb{P}}[G_H]} \cdot \mathbf{1}_A \right] \\ &= \mathbb{E}_t^{\mathbb{P}} \left[\frac{G_H}{\mathbb{E}_t^{\mathbb{P}}[G_H]} \mathbf{1}_A \right]. \end{aligned}$$

For the second property, for all $H \leq s \leq T$, given $A \in \mathcal{F}_T$, we have

$$\begin{aligned} \mathbb{R}_1^*(A | \mathcal{F}_s) &= \mathbb{E}_s^{\mathbb{R}_1^*} [\mathbf{1}_A] = \mathbb{E}_s^{\mathbb{P}} \left[\frac{\mathcal{L}_{1T}^*(H)}{\mathcal{L}_{1s}^*(H)} \mathbf{1}_A \right] = \mathbb{E}_s^{\mathbb{P}} \left[\frac{G_H / \mathbb{E}_0^{\mathbb{P}}[G_H] \cdot L_T^* / L_H^*}{G_H / \mathbb{E}_0^{\mathbb{P}}[G_H] \cdot L_s^* / L_H^*} \mathbf{1}_A \right] \\ &= \mathbb{E}_s^{\mathbb{P}} \left[\frac{L_T^*}{L_s^*} \mathbf{1}_A \right] = \mathbb{E}_s^{\mathbb{Q}^*} [\mathbf{1}_A] = \mathbb{Q}^*(A | \mathcal{F}_s). \end{aligned}$$

Furthermore, according to Proposition 1(ii), $\mathbb{R}_1^*(A | \mathcal{F}_s) = \mathbb{R}^*(A | \mathcal{F}_s)$ also holds for all $A \in \mathcal{F}_T$.

The third property is a special case of the first property when $H = T$, while the fourth property is a special case of the second property when $H = t$.

The first part of the last property is a direct result of equation (12), which was proven in equation (13). For the second part of the last property, we have

$$\mathbb{E}_t^{\mathbb{R}_1^*} [Z_T] = \mathbb{E}_t^{\mathbb{R}_1^*} \left[\mathbb{E}_H^{\mathbb{R}_1^*} [Z_T] \right] = \mathbb{E}_t^{\mathbb{R}_1^*} \left[\mathbb{E}_H^{\mathbb{Q}^*} [Z_T] \right] = \mathbb{E}_t^{\mathbb{P}} \left[\frac{G_H}{\mathbb{E}_t^{\mathbb{P}} [G_H]} \mathbb{E}_H^{\mathbb{Q}^*} [Z_T] \right],$$

where the second equality is an application of the second property, the last equality is an application of the first property, since $\mathbb{E}_H^{\mathbb{Q}^*} [Z_T]$ is a random variable that is \mathcal{F}_H measurable. \square

Propositions IA1(ii) and IA1(iv) for the \mathbb{R}_1^* measure are identical to the corresponding Propositions 1(ii) and 1(iv) for the \mathbb{R}^* measure, respectively. According to Proposition IA1(ii), the conditional \mathbb{R}_1^* probabilities at any time s on or after time H , of the events at time T , are the same as the corresponding conditional \mathbb{R}^* probabilities, as well as the conditional \mathbb{Q}^* probabilities of those events. According to Proposition IA1(iv), when $H = t$, the \mathbb{R}_1^* measure becomes the same as the \mathbb{R}^* measure and the \mathbb{Q}^* measure, and the expected value computing problem becomes the traditional claim pricing problem. Propositions IA1(i) and IA1(iii) for the \mathbb{R}_1^* measure are different from the corresponding Propositions 1(i) and 1(iii) for the \mathbb{R}^* measure, respectively. According to Proposition IA1(i), the conditional \mathbb{R}_1^* probabilities at any time t until time H , of any events at time H , are not the same as the corresponding \mathbb{P} probabilities of those events, due to an adjustment term equal to $G_H/\mathbb{E}_t^{\mathbb{P}} [G_H]$, related to the numeraire G . According to Proposition IA1(iii), when $H = T$, the \mathbb{R}_1^* measure does not become the \mathbb{P} measure, due to an adjustment term equal to $G_T/\mathbb{E}_t^{\mathbb{P}} [G_T]$, related to the numeraire G . However, there is one exception to the last argument. When the numeraire G is given as the T -maturity pure discount bond, $P(\cdot, T)$, then $G_T = P(T, T) = 1$. Therefore, for the special case of the numeraire $G = P(\cdot, T)$, Proposition IA1(iii) implies that when $H = T$, the \mathbb{R}_1^* measure becomes the \mathbb{P} measure, since $G_T/\mathbb{E}_t^{\mathbb{P}} [G_T] = P(T, T)/\mathbb{E}_t^{\mathbb{P}} [P(T, T)] = 1$. Proposition IA1(v) shows that it is easier to obtain the analytical solution of a contingent claim using the \mathbb{R}_1^* measure directly (whenever this measure leads to an analytical solution), since the term $G_H/\mathbb{E}_t^{\mathbb{P}} [G_H]$ complicates the solution using the \mathbb{R}^* measure.

B. The \mathbb{R}_2^* Measure

We first give the definition of the \mathbb{R}_2^* measure in the following lemma:

LEMMA IA1: For a fixed H with $0 \leq H \leq T$, define a process $\mathcal{L}_{2s}^*(H)$ as

$$\mathcal{L}_{2s}^*(H) = \begin{cases} \frac{G_H \mathbb{E}_s^{\mathbb{R}^*} [G_s/G_T]}{G_s \mathbb{E}_0^{\mathbb{R}^*} [G_H/G_T]} \cdot \frac{L_s^*}{L_H^*}, & \text{if } H \leq s \leq T, \\ \frac{\mathbb{E}_s^{\mathbb{R}^*} [G_H/G_T]}{\mathbb{E}_0^{\mathbb{R}^*} [G_H/G_T]}, & \text{if } 0 \leq s < H. \end{cases}$$

Let

$$\mathbb{R}_2^*(A) \triangleq \int_A \mathcal{L}_{2T}^*(H; \omega) d\mathbb{P}(\omega) \quad \text{for all } A \in \mathcal{F}_T,$$

then \mathbb{R}_2^* is a probability measure equivalent to \mathbb{P} , and $\mathcal{L}_{2s}^*(H)$ is the Radon-Nikodým derivative process of \mathbb{R}_2^* with respect to \mathbb{P} .

We omit the proof of Lemma IA1 because it follows similarly from Appendix A (Section A). For the relation between \mathbb{R}_2^* and \mathbb{R}^* , we have

$$\begin{aligned} \mathbb{E}_t^{\mathbb{R}_2^*} [Z_T] &= \mathbb{E}_t^{\mathbb{P}} \left[\frac{\mathcal{L}_{2T}^*(H)}{\mathcal{L}_{2t}^*(H)} Z_T \right] = \mathbb{E}_t^{\mathbb{P}} \left[\frac{L_T^*}{L_H^*} \cdot \frac{G_H/G_T}{\mathbb{E}_t^{\mathbb{R}^*} [G_H/G_T]} Z_T \right] \\ &= \mathbb{E}_t^{\mathbb{P}} \left[\frac{\mathcal{L}_T^*(H)}{\mathcal{L}_t^*(H)} \cdot \frac{G_H/G_T}{\mathbb{E}_t^{\mathbb{R}^*} [G_H/G_T]} Z_T \right] = \mathbb{E}_t^{\mathbb{R}^*} \left[\frac{G_H/G_T}{\mathbb{E}_t^{\mathbb{R}^*} [G_H/G_T]} Z_T \right], \end{aligned}$$

for any \mathcal{F}_T -measurable variable Z_T .

Now, we prove that the \mathbb{R}_1^* measure subsumes the \mathbb{R}_2^* measure when the numeraires are restricted to be either the money market account or the pure discount bond.

According to the classification of EEMs in Section I, when $G = B = e^{\int_0^\cdot r_u du}$; in other words, the numeraire is the money market account, we have $\mathbb{R}^* = \mathbb{R}$,

$\mathbb{R}_2^* = \mathbb{R}_2$, and $L_s^* = L_s$. Then, for \mathbb{R}_2 's Radon-Nikodým derivative process

$$\mathcal{L}_{2s}(H) = \begin{cases} \frac{B_H \mathbb{E}_s^{\mathbb{R}} [B_s/B_T]}{B_s \mathbb{E}_0^{\mathbb{R}} [B_H/B_T]} \cdot \frac{L_s}{L_H} = \frac{B_H P(s, T)}{B_s \mathbb{E}_0^{\mathbb{P}} [P(H, T)]} \cdot \frac{L_s}{L_H}, & \text{if } H \leq s \leq T, \\ \frac{\mathbb{E}_s^{\mathbb{R}} [B_H/B_T]}{\mathbb{E}_0^{\mathbb{R}} [B_H/B_T]} = \frac{\mathbb{E}_s^{\mathbb{P}} [P(H, T)]}{\mathbb{E}_0^{\mathbb{P}} [P(H, T)]}, & \text{if } 0 \leq s < H, \end{cases}$$

where $\mathbb{E}_s^{\mathbb{R}} [P(H, T)] = \mathbb{E}_s^{\mathbb{P}} [P(H, T)]$ for $0 \leq s \leq H$ due to Proposition 1.

When $G = P(\cdot, T)$, in other words, the numeraire is the pure discount bond, we have $\mathbb{R}^* = \mathbb{R}^T$, $\mathbb{R}_2^* = \mathbb{R}_2^T$, and $L_s^* = L_s^P = \frac{P(s, T)B_0}{P(0, T)B_s} L_s$. Then, for \mathbb{R}_2^T 's Radon-Nikodým derivative process

$$\mathcal{L}_{2s}^P(H) = \begin{cases} \frac{P(H, T) \mathbb{E}_s^{\mathbb{R}^T} [P(s, T)]}{P(s, T) \mathbb{E}_0^{\mathbb{R}^T} [P(H, T)]} \cdot \frac{P(s, T)B_H L_s}{P(H, T)B_s L_H} = \frac{B_H P(s, T)}{B_s \mathbb{E}_0^{\mathbb{P}} [P(H, T)]} \cdot \frac{L_s}{L_H}, & \text{if } H \leq s \leq T, \\ \frac{\mathbb{E}_s^{\mathbb{R}^T} [P(H, T)]}{\mathbb{E}_0^{\mathbb{R}^T} [P(H, T)]} = \frac{\mathbb{E}_s^{\mathbb{P}} [P(H, T)]}{\mathbb{E}_0^{\mathbb{P}} [P(H, T)]}, & \text{if } 0 \leq s < H, \end{cases}$$

where $\mathbb{E}_s^{\mathbb{R}^T} [P(H, T)] = \mathbb{E}_s^{\mathbb{P}} [P(H, T)]$ for $0 \leq s \leq H$ due to Proposition 1.

When $G = P(\cdot, T)$, we have $\mathbb{R}^* = \mathbb{R}^T$, $\mathbb{R}_1^* = \mathbb{R}_1^T$, and $L_s^* = L_s^P = \frac{P(s, T)B_0}{P(0, T)B_s} L_s$. Then, for \mathbb{R}_1^T 's Radon-Nikodým derivative process (also see Internet Appendix Section I.D)

$$\mathcal{L}_{1s}^P(H) = \begin{cases} \frac{P(H, T)}{\mathbb{E}_0^{\mathbb{P}} [P(H, T)]} \cdot \frac{P(s, T)B_H L_s}{P(H, T)B_s L_H} = \frac{B_H P(s, T)}{B_s \mathbb{E}_0^{\mathbb{P}} [P(H, T)]} \cdot \frac{L_s}{L_H}, & \text{if } H \leq s \leq T, \\ \frac{\mathbb{E}_s^{\mathbb{P}} [P(H, T)]}{\mathbb{E}_0^{\mathbb{P}} [P(H, T)]}, & \text{if } 0 \leq s < H. \end{cases}$$

As can be seen, the Radon-Nikodým derivative processes satisfy $\mathcal{L}_{1s}^P(H) = \mathcal{L}_{2s}^P(H) = \mathcal{L}_{2s}(H)$, which means $\mathbb{R}_1^T = \mathbb{R}_2^T = \mathbb{R}_2$. Hence, \mathbb{R}_1^T subsumes both \mathbb{R}_2^T and \mathbb{R}_2 measures when the numeraires are either the pure discount bond or the money market account.

C. The \mathbb{R} Measure for Multidimensional Brownian Motions

Here we consider the \mathbb{R} measure construction when there are multidimensional Brownian motions as the source of uncertainty. Assuming the existence of a risk-neutral measure \mathbb{Q} , we let the Radon-Nikodým derivative process L_s of \mathbb{Q} with respect to \mathbb{P} be

$$L_s \triangleq \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_s} = \exp \left(- \int_0^s \gamma_u dW_u^{\mathbb{P}} - \frac{1}{2} \int_0^s \|\gamma_u\|^2 du \right), \quad (\text{IA1})$$

where $\gamma_s = (\gamma_{1s}, \dots, \gamma_{Ns})$, $0 \leq s \leq T$, is an N -dimensional measurable, adapted process satisfying $\int_0^T \gamma_{iu}^2 du < \infty$ almost surely for $1 \leq i \leq N$, and the Novikov condition, that is, $\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T \|\gamma_u\|^2 du \right) \right] < \infty$. In addition, $\|\gamma_s\| = \left(\sum_{i=1}^d \gamma_{is}^2 \right)^{\frac{1}{2}}$ denotes the Euclidean norm.

Then, similarly to the case of one-dimensional Brownian motion in Appendix B, we have the following lemma:

LEMMA IA2: *In the economy coupled with multidimensional Brownian motion as the source of uncertainty, for a fixed H with $0 \leq H \leq T$, define a process $\mathcal{L}_s(H)$ as*

$$\mathcal{L}_s(H) = \exp \left(- \int_0^s \gamma_u(H) dW_u^{\mathbb{P}} - \frac{1}{2} \int_0^s \|\gamma_u(H)\|^2 du \right),$$

where

$$\gamma_s(H) \triangleq \mathbf{1}_{\{s \geq H\}} \gamma_s = \begin{cases} \gamma_s, & \text{if } H \leq s \leq T, \\ 0, & \text{if } 0 \leq s < H. \end{cases}$$

Let

$$\mathbb{R}(A) \triangleq \int_A L_T(H; \omega) d\mathbb{P}(\omega) \quad \text{for all } A \in \mathcal{F}_T.$$

Then \mathbb{R} is a probability measure equivalent to \mathbb{P} , and $\mathcal{L}_s(H)$ is the Radon-Nikodým derivative process of \mathbb{R} with respect to \mathbb{P} .

Moreover, define

$$\begin{aligned} W_s^{\mathbb{R}} &\triangleq W_s^{\mathbb{P}} + \int_0^s \gamma_u(H) du \\ &= W_s^{\mathbb{P}} + \int_0^s \mathbf{1}_{\{u \geq H\}} \gamma_u du. \end{aligned} \tag{IA2}$$

The process $W^{\mathbb{R}}$ is an N -dimensional Brownian motion under the \mathbb{R} measure.

Proof. We skip the steps that are consistent with the proof of Lemma 1. For multidimensional Brownian motions, clearly, we have

$$\begin{aligned} &\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T \|\gamma_u(H)\|^2 du \right) \right] \\ &= \mathbb{E} \left[\exp \left(\frac{1}{2} \int_H^T \|\gamma_u\|^2 du \right) \right] \leq \mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T \|\gamma_u\|^2 du \right) \right] < \infty. \end{aligned}$$

In other words, $\gamma_s(H)$ also satisfies the Novikov condition, then the defined $\mathcal{L}_s(H)$ is a martingale and $\mathbb{E}[\mathcal{L}_T(H)] = 1$. Since $\mathcal{L}_T(H)$ is defined as an exponential, it is almost surely strictly positive; hence, the probability measure \mathbb{R} is well-defined and equivalent to \mathbb{P} .

This Brownian motion transformation between \mathbb{R} and \mathbb{P} in equation (IA2) is an application of the multidimensional Girsanov theorem. The Girsanov theorem can apply to any N -dimensional process $\gamma_s(H)$ that has the properties i). $\gamma_s(H)$ is \mathcal{F}_s -adapted for each $s \in [0, T]$; ii). $\int_0^T (\gamma_{iu}(H))^2 du < \infty$ almost surely for $1 \leq i \leq N$; iii). $\gamma_s(H)$ satisfies the Novikov condition, which was proven.

For a given $0 \leq H \leq T$, by definition, when $s < H$, $\gamma_s(H) = 0$ is a constant. Since constant functions are measurable with respect to any σ -algebra, in particular with respect to \mathcal{F}_s , $s < H$, this proves that $\gamma_s(H)$ is \mathcal{F}_s -adapted for each $s < H$. When $s \geq H$, $\gamma_s(H) = \gamma_s$ is also \mathcal{F}_s -adapted by the definition of γ_s . Therefore, $\gamma_s(H)$ is \mathcal{F}_s -adapted for each $s \in [0, T]$, and condition i) is proven. Condition ii) is also satisfied due to that, for $1 \leq i \leq N$,

$$\int_0^T (\gamma_{iu}(H))^2 du = \int_0^T \mathbf{1}_{\{u \geq H\}} \gamma_{iu}^2 du = \int_H^T \gamma_{iu}^2 du \leq \int_0^T \gamma_{iu}^2 du < \infty \text{ almost surely,}$$

which ends the proof. □

D. The \mathbb{R}_1^T Measure for Multidimensional Brownian Motions

Consider the Radon-Nikodým derivative processes $\mathcal{L}_{1s}^*(H)$ and L_s^* in Lemma 2 and denote them as $\mathcal{L}_{1s}^P(H)$ and L_s^P , respectively, for the special case when the numeraire is the pure discount bond, or $G = P(\cdot, T)$. For this special case, $\mathbb{R}_1^* = \mathbb{R}_1^T$, as defined in Corollary 2. We know that L_s^P is given as

$$L_s^P \triangleq \frac{d\mathbb{Q}^T}{d\mathbb{P}} \Big|_{\mathcal{F}_s} = \frac{d\mathbb{Q}^T}{d\mathbb{Q}} \cdot \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_s} = \frac{P(s, T)B_0}{P(0, T)B_s} L_s,$$

where B is the money market account, L_s is defined in equation (B1) for one-dimensional Brownian motions, or equation (IA1) for multidimensional Brownian motions.

Assume that the numeraire bond $P(\cdot, T)$'s volatility is $v(\cdot, T)$ ², then the process of L_s^P satisfies

$$\frac{dL_s^P}{L_s^P} = (-\gamma_s - v(s, T)) dW_s^{\mathbb{P}}.$$

Then, we have the following lemma:

LEMMA IA3: *In the economy coupled with stochastic interest rates, for a fixed H with $0 \leq H \leq T$, define a process $\mathcal{L}_{1s}^P(H)$ as*

$$\mathcal{L}_{1s}^P(H) = \begin{cases} \frac{P(H, T)}{\mathbb{E}_0^{\mathbb{P}}[P(H, T)]} \cdot \frac{P(s, T)B_H}{P(H, T)B_s} \frac{L_s}{L_H} = \frac{B_H P(s, T)}{B_s \mathbb{E}_0^{\mathbb{P}}[P(H, T)]} \cdot \frac{L_s}{L_H}, & \text{if } H \leq s \leq T, \\ \frac{\mathbb{E}_s^{\mathbb{P}}[P(H, T)]}{\mathbb{E}_0^{\mathbb{P}}[P(H, T)]}, & \text{if } 0 \leq s < H. \end{cases}$$

²For example, under \mathbb{Q} , the pure discount bond follows

$$\frac{dP(s, T)}{P(s, T)} = r_s ds - v(s, T) dW_s^{\mathbb{Q}}.$$

Let

$$\mathbb{R}_1^T(A) \triangleq \int_A \mathcal{L}_{1T}^P(H; \omega) d\mathbb{P}(\omega) \quad \text{for all } A \in \mathcal{F}_T,$$

then \mathbb{R}_1^T is a probability measure equivalent to \mathbb{P} , and $\mathcal{L}_{1s}^P(H)$ is the Radon-Nikodým derivative process of \mathbb{R}_1^T with respect to \mathbb{P} .

Moreover, assume that

$$\frac{d\mathcal{L}_{1s}^P(H)}{\mathcal{L}_{1s}^P(H)} = (-\gamma_s(H) - v(s, T, H)) dW_s^{\mathbb{P}},$$

then define

$$\begin{aligned} W_s^{\mathbb{R}_1^T} &\triangleq W_s^{\mathbb{P}} + \int_0^s \gamma_u(H) du + \int_0^s v(u, T, H) du \\ &= W_s^{\mathbb{P}} + \int_0^s \mathbf{1}_{\{u \geq H\}} \gamma_u du + \int_0^s v(u, T, H) du. \end{aligned}$$

The process $W_s^{\mathbb{R}_1^T}$ is a Brownian motion under the \mathbb{R}_1^T measure.

Proof. The first part is an application of Lemma 2, and the second part follows from Proposition 26.4 in Björk (2009) with $Q^1 = \mathbb{R}_1^T$ and $Q^0 = \mathbb{P}$. \square

Notably, when $s \geq H$, $v(s, T, H)$ is equal to $v(s, T)$; when $s < H$, $v(s, T, H)$ is generally a term that is adjusted from $v(s, T)$. For example, under the Vasicek model defined in Internet Appendix Section II.A (see equation (IA3)), we have

$$v(s, T, H) = \begin{cases} \sigma_r \frac{1 - e^{-\alpha_r(T-s)}}{\alpha_r}, & \text{if } H \leq s \leq T, \\ \sigma_r \frac{e^{-\alpha_r(H-s)} - e^{-\alpha_r(T-s)}}{\alpha_r}, & \text{if } 0 \leq s < H. \end{cases}$$

II. Expected Future Bond Prices under Specific ATSMs

A. Expected Future Bond Price under the Vasicek Model

For the ease of exposition, this section illustrates the solution of the expected future price of a pure discount bond under the classical one-factor model of Vasicek (1977). The Ornstein-Uhlenbeck process for the short rate under the Vasicek model is given as follows:

$$dr_s = \alpha_r (m_r - r_s) ds + \sigma_r dW_s^{\mathbb{P}}, \quad (\text{IA3})$$

where α_r is the speed of mean reversion, m_r is the long-term mean of the short rate, and σ_r is the volatility of the changes in the short rate. Assuming $W_s^{\mathbb{Q}} = W_s^{\mathbb{P}} + \int_0^s \gamma_r du$, the short rate process under the \mathbb{R} measure can be derived using Lemma 1 and Internet Appendix Section I.C, as follows:

$$\begin{aligned} dr_s &= [\alpha_r (m_r - r_s) - \mathbb{1}_{\{s \geq H\}} \sigma_r \gamma_r] ds + \sigma_r dW_s^{\mathbb{R}} \\ &= [\alpha_r (m_r - r_s) \mathbb{1}_{\{s < H\}} + \alpha_r^* (m_r^* - r_s) \mathbb{1}_{\{s \geq H\}}] ds + \sigma_r dW_s^{\mathbb{R}}, \end{aligned}$$

where $\alpha_r^* = \alpha_r$, $m_r^* = m_r - \sigma_r \gamma_r / \alpha_r$.

The expected future price of a T -maturity pure discount bond at time H is given under the Vasicek model as follows: For all $t \leq H \leq T$,

$$\begin{aligned} \mathbb{E}_t [P(H, T)] &= \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T r_u du \right) \right] \\ &= \exp \left(-A_{\alpha_r^*, m_r^*}^{(b^*, c^*)}(T - H) - A_{\alpha_r, m_r}^{(b, c)}(H - t) - B_{\alpha_r, m_r}^{(b, c)}(H - t) \cdot r_t \right), \end{aligned} \quad (\text{IA4})$$

where $b^* = 0$, $c^* = 1$, $b = B_{\alpha_r^*, m_r^*}^{(b^*, c^*)}(T - H)$, $c = 0$. For any well-behaved b , c , α , and

m , $A_{\alpha,m}^{(b,c)}(\tau)$ and $B_{\alpha,m}^{(b,c)}(\tau)$ are given by

$$\begin{aligned} A_{\alpha,m}^{(b,c)}(\tau) &= \alpha m \left(bB_{\alpha}(\tau) + c \frac{\tau - B_{\alpha}(\tau)}{\alpha} \right) \\ &\quad - \frac{1}{2} \sigma_r^2 \left(b^2 \frac{1 - e^{-2\alpha\tau}}{2\alpha} + bcB_{\alpha}(\tau)^2 + c^2 \frac{2\tau - 2B_{\alpha}(\tau) - \alpha B_{\alpha}(\tau)^2}{2\alpha^2} \right), \\ B_{\alpha,m}^{(b,c)}(\tau) &= be^{-\alpha\tau} + cB_{\alpha}(\tau), \end{aligned}$$

with $B_{\alpha}(\tau) = (1 - e^{-\alpha\tau})/\alpha$. When $H = t$, equation (IA4) reduces to the Vasicek bond price solution (see Vasicek, 1977, equation 27).

B. Expected Future Bond Price under the CIR Model

We now illustrate the expected future price of a pure discount bond under the classic one-factor Cox-Ingersoll-Ross (CIR, Cox et al., 1985) model. The square-root process for the short rate under the CIR model is given as follows:

$$dr_s = \alpha_r (m_r - r_s) ds + \sigma_r \sqrt{r_s} dW_s^{\mathbb{P}}.$$

Assume that the market price of risk γ_s satisfies

$$\gamma_s = \gamma_r \sqrt{r_s}.$$

Thus, $W_s^{\mathbb{Q}} = W_s^{\mathbb{P}} + \int_0^s \gamma_r \sqrt{r_u} du$, and the short rate process under the \mathbb{R} measure can be derived using Lemma 1 and Internet Appendix Section I.C, as follows:

$$\begin{aligned} dr_s &= [\alpha_r (m_r - r_s) - \mathbb{1}_{\{s \geq H\}} \gamma_r \sigma_r r_s] ds + \sigma_r \sqrt{r_s} dW_s^{\mathbb{R}} \\ &= [\alpha_r (m_r - r_s) \mathbb{1}_{\{s < H\}} + \alpha_r^* (m_r^* - r_s) \mathbb{1}_{\{s \geq H\}}] ds + \sigma_r \sqrt{r_s} dW_s^{\mathbb{R}}, \end{aligned}$$

where $\alpha_r^* = \alpha_r + \gamma_r \sigma_r$, $m_r^* = \alpha_r m_r / (\alpha_r + \gamma_r \sigma_r)$.

The expected future price of a T -maturity pure discount bond at time H is given under the CIR model as follows: For all $t \leq H \leq T$,

$$\mathbb{E}_t [P(H, T)] = \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T r_u du \right) \right]$$

$$= \exp \left(-A_{\alpha_r^*, m_r^*}^{(b^*, c^*)}(T - H) - A_{\alpha_r, m_r}^{(b, c)}(H - t) - B_{\alpha_r, m_r}^{(b, c)}(H - t) \cdot r_t \right), \quad (\text{IA5})$$

where $b^* = 0$, $c^* = 1$, $b = B_{\alpha_r^*, m_r^*}^{(b^*, c^*)}(T - H)$, $c = 0$. For any well-behaved b , c , α , and m , $A_{\alpha, m}^{(b, c)}(\tau)$ and $B_{\alpha, m}^{(b, c)}(\tau)$ are given as

$$A_{\alpha, m}^{(b, c)}(\tau) = -\frac{2\alpha m}{\sigma_r^2} \ln \left(\frac{2\beta e^{\frac{1}{2}(\beta + \alpha)\tau}}{b\sigma_r^2(e^{\beta\tau} - 1) + \beta - \alpha + e^{\beta\tau}(\beta + \alpha)} \right),$$

$$B_{\alpha, m}^{(b, c)}(\tau) = \frac{b(\beta + \alpha + e^{\beta\tau}(\beta - \alpha)) + 2c(e^{\beta\tau} - 1)}{b\sigma_r^2(e^{\beta\tau} - 1) + \beta - \alpha + e^{\beta\tau}(\beta + \alpha)},$$

with $\beta = \sqrt{\alpha^2 + 2\sigma_r^2 c}$. When $H = t$, equation (IA5) reduces to the CIR bond price solution (see Cox et al., 1985, equation 23).

C. Expected Future Bond Price under the $A_{1r}(3)$ Model

Here, we employ the commonly used $A_{1r}(3)$ form of three-factor ATSMs, which has been developed in various models, including Balduzzi et al. (1996), Dai and Singleton (2000). Specifically, we use the $A_{1r}(3)_{\text{MAX}}$ model to compute the expected bond price. Under the physical measure \mathbb{P} , the state variable processes under the $A_{1r}(3)_{\text{MAX}}$ model are defined as follows:

$$\begin{aligned} dv_s &= \alpha_v(m_v - v_s)ds + \eta\sqrt{v_s}dW_{1s}^{\mathbb{P}}, \\ d\theta_s &= \alpha_\theta(m_\theta - \theta_s)ds + \sigma_{\theta v}\eta\sqrt{v_s}dW_{1s}^{\mathbb{P}} + \sqrt{\zeta^2 + \beta_\theta v_s}dW_{2s}^{\mathbb{P}} + \sigma_{\theta r}\sqrt{\delta_r + v_s}dW_{3s}^{\mathbb{P}}, \\ dr_s &= \alpha_{rv}(m_v - v_s)ds + \alpha_r(\theta_s - r_s)ds + \sigma_{rv}\eta\sqrt{v_s}dW_{1s}^{\mathbb{P}} + \sigma_{r\theta}\sqrt{\zeta^2 + \beta_\theta v_s}dW_{2s}^{\mathbb{P}} \\ &\quad + \sqrt{\delta_r + v_s}dW_{3s}^{\mathbb{P}}, \end{aligned}$$

where $W_1^{\mathbb{P}}$, $W_2^{\mathbb{P}}$ and $W_3^{\mathbb{P}}$ are independent Brownian motions under the physical measure \mathbb{P} . We assume the market price of risks (MPRs), $\gamma_s = (\gamma_{1s}, \gamma_{2s}, \gamma_{3s})$, are given by

$$\begin{aligned} \gamma_{1s} &= \gamma_1\sqrt{v_s}, \\ \gamma_{2s} &= \gamma_2\sqrt{\zeta^2 + \beta_\theta v_s}, \\ \gamma_{3s} &= \gamma_3\sqrt{\delta_r + v_s}. \end{aligned}$$

Therefore, using Lemma 1 and Internet Appendix Section I.C, the state variable processes under the $A_{1r}(3)_{\text{MAX}}$ model under the \mathbb{R} measure are given by

$$\begin{aligned} dv_s &= [\alpha_v (m_v - v_s) - \mathbb{1}_{\{s \geq H\}} \gamma_1 \eta v_s] ds + \eta \sqrt{v_s} dW_{1s}^{\mathbb{R}}, \\ d\theta_s &= [\alpha_\theta (m_\theta - \theta_s) - \mathbb{1}_{\{s \geq H\}} \gamma_1 \sigma_{\theta v} \eta v_s - \mathbb{1}_{\{s \geq H\}} \gamma_2 (\zeta^2 + \beta_\theta v_s) \\ &\quad - \mathbb{1}_{\{s \geq H\}} \gamma_3 \sigma_{\theta r} (\delta_r + v_s)] ds + \sigma_{\theta v} \eta \sqrt{v_s} dW_{1s}^{\mathbb{R}} + \sqrt{\zeta^2 + \beta_\theta v_s} dW_{2s}^{\mathbb{R}} + \sigma_{\theta r} \sqrt{\delta_r + v_s} dW_{3s}^{\mathbb{R}}, \\ dr_s &= [\alpha_{rv} (m_v - v_s) + \alpha_r (\theta_s - r_s) - \mathbb{1}_{\{s \geq H\}} \gamma_1 \sigma_{rv} \eta v_s - \mathbb{1}_{\{s \geq H\}} \gamma_2 \sigma_{r\theta} (\zeta^2 + \beta_\theta v_s) \\ &\quad - \mathbb{1}_{\{s \geq H\}} \gamma_3 (\delta_r + v_s)] ds + \sigma_{rv} \eta \sqrt{v_s} dW_{1s}^{\mathbb{R}} + \sigma_{r\theta} \sqrt{\zeta^2 + \beta_\theta v_s} dW_{2s}^{\mathbb{R}} + \sqrt{\delta_r + v_s} dW_{3s}^{\mathbb{R}}. \end{aligned}$$

Hence, using equation (15) in Corollary 1, the time t expectation of the future price of a T -maturity default-free zero-coupon bond at time H under the $A_{1r}(3)_{\text{MAX}}$ model is given as follows. For all $t \leq H \leq T$,

$$\begin{aligned} \mathbb{E}_t [P(H, T)] &= \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T r_u du \right) \right] \\ &= \exp \left(-A_{\chi^*} \begin{pmatrix} \lambda_1^*, \mu_1^* \\ \lambda_2^*, \mu_2^* \\ \lambda_3^*, \mu_3^* \end{pmatrix} (T - H) - A_{\chi} \begin{pmatrix} \lambda_1, \mu_1 \\ \lambda_2, \mu_2 \\ \lambda_3, \mu_3 \end{pmatrix} (H - t) - B_{\chi} \begin{pmatrix} \lambda_1, \mu_1 \\ \lambda_2, \mu_2 \\ \lambda_3, \mu_3 \end{pmatrix} (H - t) v_t \right. \\ &\quad \left. - C_{\chi} \begin{pmatrix} \lambda_1, \mu_1 \\ \lambda_2, \mu_2 \\ \lambda_3, \mu_3 \end{pmatrix} (H - t) \theta_t - D_{\chi} \begin{pmatrix} \lambda_1, \mu_1 \\ \lambda_2, \mu_2 \\ \lambda_3, \mu_3 \end{pmatrix} (H - t) r_t \right), \end{aligned}$$

where

$$\begin{aligned} \lambda_1^* &= 0, \quad \lambda_2^* = 0, \quad \lambda_3^* = 0, \quad \mu_1^* = 0, \quad \mu_2^* = 0, \quad \mu_3^* = 1, \quad \mu_1 = 0, \quad \mu_2 = 0, \quad \mu_3 = 0, \\ \lambda_1 &= B_{\chi^*} \begin{pmatrix} \lambda_1^*, \mu_1^* \\ \lambda_2^*, \mu_2^* \\ \lambda_3^*, \mu_3^* \end{pmatrix} (T - H), \quad \lambda_2 = C_{\chi^*} \begin{pmatrix} \lambda_1^*, \mu_1^* \\ \lambda_2^*, \mu_2^* \\ \lambda_3^*, \mu_3^* \end{pmatrix} (T - H), \quad \lambda_3 = D_{\chi^*} \begin{pmatrix} \lambda_1^*, \mu_1^* \\ \lambda_2^*, \mu_2^* \\ \lambda_3^*, \mu_3^* \end{pmatrix} (T - H), \end{aligned}$$

and

$$\begin{aligned} \chi &= (\alpha_v, \alpha_\theta, \alpha_r, \alpha_{rv}, m_v, m_\theta, \sigma_{\theta v}, \sigma_{\theta r}, \sigma_{rv}, \sigma_{r\theta}, \eta, \zeta, \beta_\theta, \delta_r, 0, 0), \\ \chi^* &= (\alpha_v^*, \alpha_\theta^*, \alpha_r^*, \alpha_{rv}^*, m_v^*, m_\theta^*, \sigma_{\theta v}, \sigma_{\theta r}, \sigma_{rv}, \sigma_{r\theta}, \eta, \zeta, \beta_\theta, \delta_r, \alpha_{\theta v}^*, m_r^*), \end{aligned}$$

with

$$\begin{aligned}\alpha_v^* &= \alpha_v + \gamma_1 \eta, & m_v^* &= \frac{\alpha_v m_v}{\alpha_v + \gamma_1 \eta}, & \alpha_\theta^* &= \alpha_\theta, & m_\theta^* &= \frac{\alpha_\theta m_\theta - \gamma_2 \zeta^2 - \gamma_3 \sigma_{\theta r} \delta_r}{\alpha_\theta}, \\ \alpha_r^* &= \alpha_r, & \alpha_{rv}^* &= \alpha_{rv} + \gamma_1 \sigma_{rv} \eta + \gamma_2 \sigma_{r\theta} \beta_\theta + \gamma_3, & \alpha_{\theta v}^* &= -\gamma_1 \sigma_{\theta v} \eta - \gamma_2 \beta_\theta - \gamma_3 \sigma_{\theta r}, \\ m_r^* &= \alpha_{rv} m_v - \alpha_{rv}^* m_v^* - \gamma_2 \sigma_{r\theta} \zeta^2 - \gamma_3 \delta_r.\end{aligned}$$

For any well-defined λ_i, μ_i ($i = 1, 2, 3$) and the characteristic χ , $A_\chi(\tau) \triangleq A_\chi^{\binom{\lambda_1, \mu_1}{\lambda_2, \mu_2}{\lambda_3, \mu_3}}(\tau)$, $B_\chi(\tau) \triangleq B_\chi^{\binom{\lambda_1, \mu_1}{\lambda_2, \mu_2}{\lambda_3, \mu_3}}(\tau)$, $C_\chi(\tau) \triangleq C_\chi^{\binom{\lambda_1, \mu_1}{\lambda_2, \mu_2}{\lambda_3, \mu_3}}(\tau)$, $D_\chi(\tau) \triangleq D_\chi^{\binom{\lambda_1, \mu_1}{\lambda_2, \mu_2}{\lambda_3, \mu_3}}(\tau)$ satisfy the following ODEs

$$\begin{aligned}\frac{\partial A_\chi(\tau)}{\partial \tau} &= B_\chi(\tau) \alpha_v m_v + C_\chi(\tau) \alpha_\theta m_\theta + D_\chi(\tau) (\alpha_{rv} m_v + m_r) - \frac{1}{2} C_\chi^2(\tau) (\sigma_{\theta r}^2 \delta_r + \zeta^2) \\ &\quad - \frac{1}{2} D_\chi^2(\tau) (\sigma_{r\theta}^2 \zeta^2 + \delta_r) - C_\chi(\tau) D_\chi(\tau) (\sigma_{r\theta} \zeta^2 + \sigma_{\theta r} \delta_r), \\ \frac{\partial B_\chi(\tau)}{\partial \tau} &= -B_\chi(\tau) \alpha_v + C_\chi(\tau) \alpha_{\theta v} - D_\chi(\tau) \alpha_{rv} - \frac{1}{2} B_\chi^2(\tau) \eta^2 - \frac{1}{2} C_\chi^2(\tau) (\beta_\theta + \sigma_{\theta v}^2 \eta^2 + \sigma_{\theta r}^2) \\ &\quad - \frac{1}{2} D_\chi^2(\tau) (\sigma_{rv}^2 \eta^2 + \sigma_{r\theta}^2 \beta_\theta + 1) - B_\chi(\tau) C_\chi(\tau) \sigma_{\theta v} \eta^2 - B_\chi(\tau) D_\chi(\tau) \sigma_{rv} \eta^2 \\ &\quad - C_\chi(\tau) D_\chi(\tau) (\sigma_{\theta v} \sigma_{rv} \eta^2 + \sigma_{r\theta} \beta_\theta + \sigma_{\theta r}) + \mu_1, \\ \frac{\partial C_\chi(\tau)}{\partial \tau} &= -C_\chi(\tau) \alpha_\theta + D_\chi(\tau) \alpha_r + \mu_2, \\ \frac{\partial D_\chi(\tau)}{\partial \tau} &= -D_\chi(\tau) \alpha_r + \mu_3,\end{aligned}\tag{IA6}$$

subject to the boundary condition $A_\chi = 0$, $B_\chi(0) = \lambda_1$, $C_\chi(0) = \lambda_2$, $D_\chi(0) = \lambda_3$. $A_\chi(\tau)$ and $B_\chi(\tau)$ can be solved easily using numerical integration, and $C_\chi(\tau)$ and $D_\chi(\tau)$ are given as

$$\begin{aligned}C_\chi(\tau) &= \lambda_2 e^{-\alpha_\theta \tau} + \frac{\mu_2 + \mu_3}{\alpha_\theta} (1 - e^{-\alpha_\theta \tau}) + \frac{\lambda_3 \alpha_r - \mu_3}{\alpha_\theta - \alpha_r} (e^{-\alpha_r \tau} - e^{-\alpha_\theta \tau}), \\ D_\chi(\tau) &= \lambda_3 e^{-\alpha_r \tau} + \frac{\mu_3}{\alpha_r} (1 - e^{-\alpha_r \tau}).\end{aligned}$$

Similarly, $A_{\chi^*}(\tau) \triangleq A_{\chi^*}^{\begin{pmatrix} \lambda_1^*, \mu_1^* \\ \lambda_2^*, \mu_2^* \\ \lambda_3^*, \mu_3^* \end{pmatrix}}(\tau)$, $B_{\chi^*}(\tau) \triangleq B_{\chi^*}^{\begin{pmatrix} \lambda_1^*, \mu_1^* \\ \lambda_2^*, \mu_2^* \\ \lambda_3^*, \mu_3^* \end{pmatrix}}(\tau)$, $C_{\chi^*}(\tau) \triangleq C_{\chi^*}^{\begin{pmatrix} \lambda_1^*, \mu_1^* \\ \lambda_2^*, \mu_2^* \\ \lambda_3^*, \mu_3^* \end{pmatrix}}(\tau)$, $D_{\chi^*}(\tau) \triangleq D_{\chi^*}^{\begin{pmatrix} \lambda_1^*, \mu_1^* \\ \lambda_2^*, \mu_2^* \\ \lambda_3^*, \mu_3^* \end{pmatrix}}(\tau)$ are obtained as in equation (IA6) with the characteristic χ replaced with the characteristic χ^* , and the parameters λ_i, μ_i ($i = 1, 2, 3$) replaced with λ_i^*, μ_i^* ($i = 1, 2, 3$).

III. Expected Future Bond Price under the Quadratic Term Structure Model

As in Ahn et al. (2002), an N -factor quadratic term structure model (QTSM) satisfies the following assumptions. First, the nominal instantaneous interest rate is a quadratic function of the state variables

$$r_s = \alpha + \beta'Y_s + Y_s'\Psi Y_s,$$

where α is a constant, β is an N -dimensional vector, and Ψ is an $N \times N$ matrix of constants. We assume that $\alpha - \frac{1}{4}\beta'\Psi^{-1}\beta \geq 0$, and Ψ is a positive semidefinite matrix.

The SDEs of the state variables Y are characterized as multivariate Gaussian processes with mean reverting properties:

$$dY_s = (\mu + \xi Y_s) ds + \Sigma dW_s^{\mathbb{P}},$$

where μ is an N -dimensional vector of constants and ξ and Σ are N -dimensional square matrices. We assume that ξ is “diagonalizable” and has negative real components of eigenvalues. $W^{\mathbb{P}}$ is an N -dimensional vector of standard Brownian motions that are mutually independent.

Similar to Ahn et al. (2002), the market price of risks γ_s is assumed to be

$$\gamma_s = \Sigma^{-1}(\gamma_0 + \gamma_1 Y_s),$$

where γ_0 is an N -dimensional vector, γ_1 is an $N \times N$ matrix satisfying that

$\Sigma^{-1}\Sigma^{-1}\gamma_1$ is symmetric. Hence, using Lemma 1 and Internet Appendix Section I.C, the state variables Y under the \mathbb{R} measure satisfy the following SDE:

$$\begin{aligned} dY_s &= [\mu + \xi Y_s - \mathbf{1}_{\{s \geq H\}} (\gamma_0 + \gamma_1 Y_s)] ds + \Sigma dW_s^{\mathbb{R}} \\ &= [(\mu + \xi Y_s) \mathbf{1}_{\{s < H\}} + (\mu^* + \xi^* Y_s) \mathbf{1}_{\{s \geq H\}}] ds + \Sigma dW_s^{\mathbb{R}}, \end{aligned}$$

where $\mu^* = \mu - \gamma_0$, $\xi^* = \xi - \gamma_1$.

Therefore, the time t expectation of the future price of a T -maturity default-free zero-coupon bond at time H under the N -factor QTSM model is given as follows. For all $t \leq H \leq T$,

$$\begin{aligned} \mathbb{E}_t [P(H, T)] &= \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T r_u du \right) \right] = \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T (\alpha + \beta' Y_u + Y_u' \Psi Y_u) du \right) \right] \\ &= \exp \left(-\alpha(T-H) - A_{\mu^*, \xi^*}^{(b^*, d^*)}(T-H) - A_{\mu, \xi}^{(b, d)}(H-t) - B_{\mu, \xi}^{(b, d)}(H-t)' Y_t \right. \\ &\quad \left. - Y_t' C_{\mu, \xi}^{(b, d)}(H-t) Y_t \right), \end{aligned}$$

where

$$\begin{aligned} b^* &= 0, \quad c^* = 0, \quad d^* = \beta, \quad q^* = \Psi, \\ b &= B_{\mu^*, \xi^*}^{(b^*, d^*)}(T-H), \quad c = C_{\mu^*, \xi^*}^{(b^*, d^*)}(T-H), \quad d = 0, \quad q = 0. \end{aligned}$$

For any well-defined matrices b, c, d, q, μ and ξ , $A_{\mu, \xi}(\tau) \triangleq A_{\mu, \xi}^{(b, d)}(\tau)$, $B_{\mu, \xi}(\tau) \triangleq B_{\mu, \xi}^{(b, d)}(\tau)$, and $C_{\mu, \xi}(\tau) \triangleq C_{\mu, \xi}^{(b, d)}(\tau)$ satisfy the following ODEs

$$\begin{aligned} \frac{dA_{\mu, \xi}(\tau)}{d\tau} &= \mu' B_{\mu, \xi}(\tau) - \frac{1}{2} B_{\mu, \xi}(\tau)' \Sigma \Sigma' B_{\mu, \xi}(\tau) + \text{tr}(\Sigma \Sigma' C_{\mu, \xi}(\tau)), \\ \frac{dB_{\mu, \xi}(\tau)}{d\tau} &= \xi' B_{\mu, \xi}(\tau) + 2C_{\mu, \xi}(\tau) \mu - 2C_{\mu, \xi}(\tau) \Sigma \Sigma' B_{\mu, \xi}(\tau) + d, \quad (\text{IA7}) \\ \frac{dC_{\mu, \xi}(\tau)}{d\tau} &= -2C_{\mu, \xi}(\tau) \Sigma \Sigma' C_{\mu, \xi}(\tau) + C_{\mu, \xi}(\tau) \xi + \xi' C_{\mu, \xi}(\tau) + q, \end{aligned}$$

with the terminal conditions $A_{\mu, \xi}(0) = 0$, $B_{\mu, \xi}(0) = b$, $C_{\mu, \xi}(0) = c$. These ODEs

can be easily solved numerically. $A_{\mu^*, \xi^*}(\tau) \triangleq A_{\mu^*, \xi^*}^{(b^*, d^*)}(\tau)$, $B_{\mu^*, \xi^*}(\tau) \triangleq B_{\mu, \xi}^{(b^*, d^*)}(\tau)$, and $C_{\mu^*, \xi^*}(\tau) \triangleq C_{\mu^*, \xi^*}^{(c^*, q^*)}(\tau)$ are obtained similarly as in equation (IA7) with the drift coefficients (μ, ξ) replaced with drift coefficient (μ^*, ξ^*) , and the parameters (b, c, d, q) replaced with (b^*, c^*, d^*, q^*) .

Example: The QTSM3 model. Since the empirical literature suggests that three factors are required to describe the term structure (see, e.g., Knez et al., 1996; Dai and Singleton, 2000), we fix $N = 3$ and explore the particular QTSM3 model defined in Ahn et al. (2002). According to the parameter restrictions by Ahn et al. (2002), in this case, $\mu \geq 0$, $\alpha > 0$, $\beta = 0$, and Σ is a diagonal matrix. In addition, ξ and γ_1 are diagonal, and this assumption results in orthogonal state variables under the \mathbb{P} measure as well as the \mathbb{R} measure. Furthermore, Ψ is the 3×3 identity matrix I_3 , such that there are no interactions among the state variables in the determination of the interest rate. The number of parameters in QTSM3 is 16. An important advantage is that the QTSM3 model allows for a fully closed-form solution for the expected bond price:

$$\begin{aligned} \mathbb{E}_t [P(H, T)] &= \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T r_u du \right) \right] = \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T (\alpha + Y_u' \Psi Y_u) du \right) \right] \\ &= \exp \left(-\alpha(T - H) - \sum_{i=1}^3 A_{\mu_i^*, \xi_{ii}^*}^{(b_i^*, 0)}(T - H) - \sum_{i=1}^3 A_{\mu_i, \xi_{ii}}^{(b_i, 0)}(H - t) \right. \\ &\quad \left. - \sum_{i=1}^3 B_{\mu_i, \xi_{ii}}^{(b_i, 0)}(H - t) Y_{it} - \sum_{i=1}^3 C_{\mu_i, \xi_{ii}}^{(b_i, 0)}(H - t) Y_{it}^2 \right), \quad (\text{IA8}) \end{aligned}$$

where $\mu_i^* = \mu_i - \gamma_{0i}$, $\xi_{ii}^* = \xi_{ii} - \gamma_{1ii}$, and

$$b_i^* = 0, \quad c_i^* = 0, \quad q_i^* = 1, \quad b_i = B_{\mu_i^*, \xi_{ii}^*}^{(b_i^*, 0)}(T - H), \quad c_i = C_{\mu_i^*, \xi_{ii}^*}^{(c_i^*, q_i^*)}(T - H), \quad q_i = 0.$$

For any well-defined b_i, c_i, q_i , and the coefficients μ_i, ξ_{ii} , we have

$$\begin{aligned}
A_{\mu_i, \xi_{ii}}^{(b_i, 0)}(\tau) &= \frac{\left(\frac{\mu_i}{\beta_i}\right)^2 q_i \tau - \frac{b_i^2 \beta_i \Sigma_{ii}^2 (e^{2\beta_i \tau} - 1) - 2b_i \mu_i (e^{\beta_i \tau} - 1) [(\beta_i - \xi_{ii}) (e^{\beta_i \tau} - 1) + 2\beta_i]}{2\beta_i [(2c_i \Sigma_{ii}^2 + \beta_i - \xi_{ii}) (e^{2\beta_i \tau} - 1) + 2\beta_i]} \\
&\quad - \frac{\mu_i^2 (e^{\beta_i \tau} - 1) [((\beta_i - 2\xi_{ii}) q_i - 2c_i \xi_{ii}^2) (e^{\beta_i \tau} - 1) + 2\beta_i q_i]}{\beta_i^3 [(2c_i \Sigma_{ii}^2 + \beta_i - \xi_{ii}) (e^{2\beta_i \tau} - 1) + 2\beta_i]} \\
&\quad - \frac{1}{2} \ln \frac{2\beta_i e^{(\beta_i - \xi_{ii}) \tau}}{(2c_i \Sigma_{ii}^2 + \beta_i - \xi_{ii}) (e^{2\beta_i \tau} - 1) + 2\beta_i}, \\
B_{\mu_i, \xi_{ii}}^{(b_i, 0)}(\tau) &= \frac{2b_i \beta_i^2 e^{\beta_i \tau} + 2\mu_i [(q_i + c_i (\xi_{ii} + \beta_i)) (e^{\beta_i \tau} - 1)^2 + 2c_i \beta_i (e^{\beta_i \tau} - 1)]}{\beta_i [(2c_i \Sigma_{ii}^2 + \beta_i - \xi_{ii}) (e^{2\beta_i \tau} - 1) + 2\beta_i]}, \\
C_{\mu_i, \xi_{ii}}^{(b_i, 0)}(\tau) &= \frac{c_i [2\beta_i + (e^{2\beta_i \tau} - 1) (\beta_i + \xi_{ii})] + q_i (e^{2\beta_i \tau} - 1)}{(2c_i \Sigma_{ii}^2 + \beta_i - \xi_{ii}) (e^{2\beta_i \tau} - 1) + 2\beta_i},
\end{aligned} \tag{IA9}$$

where $\beta_i = \sqrt{\xi_{ii}^2 + 2\Sigma_{ii}^2 q_i}$. $A_{\mu_i^*, \xi_{ii}^*}^{(b_i^*, 0)}(\tau)$, $B_{\mu_i^*, \xi_{ii}^*}^{(b_i^*, 0)}(\tau)$ and $C_{\mu_i^*, \xi_{ii}^*}^{(b_i^*, 0)}(\tau)$ are obtained similarly as in equation (IA9) with the coefficients (μ_i, ξ_{ii}) replaced with the coefficients (μ_i^*, ξ_{ii}^*) , and the parameters (b_i, c_i, q_i) replaced with the parameters (b_i^*, c_i^*, q_i^*) .

IV. Expected Future Price of Forward-Start Option using the Forward-Start R -Transform

As an extension of the R -transforms in Section II.C, we propose the following *forward-start R -transform*, which can be used to obtain the expected future price of a forward-start option with the terminal date T payoff given as $(S_T - kS_{T_0})^+$, with T_0 as the intermediate future date on which the strike price kS_{T_0} is determined:

$$\psi^R(a_1, a_2, z) \triangleq \mathbb{E}_t^{\mathbb{R}} \left[\exp \left(- \int_H^T r_u du + a_1 \ln S_T + a_2 \ln S_{T_0} + z (\ln S_T - \ln S_{T_0}) \right) \right].$$

The analytical solution of the above forward-start R -transform can be derived under the affine-jump diffusion processes given in Duffie et al. (2000) using the \mathbb{R}

measure.³ The solution of $\psi^R(a_1, a_2, z)$ can be used to obtain the expected future price of the forward-start call option as follows:

$$\begin{aligned}\mathbb{E}_t[C_H] &= \mathbb{E}_t^{\mathbb{R}} \left[e^{-\int_H^T r_u du} (S_T - kS_{T_0})^+ \right] \\ &= \mathbb{E}_t^{\mathbb{R}} \left[e^{-\int_H^T r_u du} S_T \mathbb{1}_{\{\ln S_T - \ln S_{T_0} > \ln k\}} \right] - k \mathbb{E}_t^{\mathbb{R}} \left[e^{-\int_H^T r_u du} S_{T_0} \mathbb{1}_{\{\ln S_T - \ln S_{T_0} > \ln k\}} \right] \\ &= \mathbb{E}_t^{\mathbb{R}} \left[e^{-\int_H^T r_u du} S_T \right] \Pi_1 - k \mathbb{E}_t^{\mathbb{R}} \left[e^{-\int_H^T r_u du} S_{T_0} \right] \Pi_2.\end{aligned}$$

Using the Fourier inversion method outlined in Bakshi and Madan (2000) and Duffie et al. (2000), the expected future price of the forward-start option can be solved as

$$\mathbb{E}_t[C_H] = \mathbb{E}_t^{\mathbb{R}} \left[e^{-\int_H^T r_u du} S_T \right] \Pi_1 - k \mathbb{E}_t^{\mathbb{R}} \left[e^{-\int_H^T r_u du} S_{T_0} \right] \Pi_2,$$

where

$$\mathbb{E}_t^{\mathbb{R}} \left[e^{-\int_H^T r_u du} S_T \right] = \psi^R(1, 0, 0) \quad \text{and} \quad \mathbb{E}_t^{\mathbb{R}} \left[e^{-\int_H^T r_u du} S_{T_0} \right] = \psi^R(0, 1, 0),$$

and for $j = 1, 2$, the two probabilities Π_j are calculated by using the Fourier inversion formula:

$$\Pi_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[\frac{e^{-iu \ln k} \Phi^j(u)}{iu} \right] du,$$

with the corresponding characteristic functions $\Phi^1(u)$ and $\Phi^2(u)$, given as

$$\Phi^1(u) = \frac{\psi^R(1, 0, iu)}{\psi^R(1, 0, 0)} \quad \text{and} \quad \Phi^2(u) = \frac{\psi^R(0, 1, iu)}{\psi^R(0, 1, 0)}.$$

³This solution also extends the work of Kruse and Nögel (2005) for computing the *expected future price* of the forward-start call option under the affine-jump diffusion processes given in Duffie et al. (2000). There exist two solutions of $\psi^R(a_1, a_2, z)$ depending upon whether $t \leq H \leq T_0 \leq T$ or $t \leq T_0 \leq H \leq T$.

V. Expected Future Option Price under the Merton Model with Vasicek Short Rate Process

Consider a special case of equations (36) and (37) with $\sigma(s) = \sigma$, $\mu(s) = r_s + \gamma\sigma$, $\rho(s) = \rho$, with the parameters σ and ρ , given as constants, and $\gamma = (\mu(s) - r_s) / \sigma$ (defined as the asset's risk premium per unit of asset return volatility), also given as a constant. Assume that the bond price process is given by the single-factor Vasicek model (see Internet Appendix Section II.A). The asset price process, the short rate process, and the bond price process under these specifications are given as:

$$\begin{aligned} \frac{dS_s}{S_s} &= (r_s + \gamma\sigma) ds + \sigma dW_{1s}^{\mathbb{P}}, \\ dr_s &= \alpha_r (m_r - r_s) ds + \sigma_r dW_{2s}^{\mathbb{P}}, \\ \frac{dP(s, T)}{P(s, T)} &= (r_s - \gamma_r \sigma_r B_{\alpha_r}(T - s)) ds - \sigma_r B_{\alpha_r}(T - s) dW_{2s}^{\mathbb{P}}, \end{aligned} \tag{IA10}$$

where $B_{\alpha_r}(\tau) = (1 - e^{-\alpha_r \tau}) / \alpha_r$, and γ_r is the market price of interest rate risk in the Vasicek (1977) model.

The full solution of the expected call price formula given by equation (40) can be obtained by solving $\mathbb{E}_t[S_H]$, $\mathbb{E}_t[P(H, T)]$, and v_p for the above processes. Using Lemma 2 and Internet Appendix Section I.D, the normalized asset price $V = S/P(\cdot, T)$ follows

$$\begin{aligned} \frac{dV_s}{V_s} &= \mathbf{1}_{\{s < H\}} [\gamma\sigma + \gamma_r \sigma_r B_{\alpha_r}(T - s) + (\rho\sigma + \sigma_r B_{\alpha_r}(T - s)) \sigma_r B_{\alpha_r}(H - s)] ds \\ &\quad + \sigma dW_{1s}^{\mathbb{R}_1^T} + \sigma_r B_{\alpha_r}(T - s) dW_{2s}^{\mathbb{R}_1^T}. \end{aligned}$$

We can see that V_T is also lognormally distributed under the \mathbb{R}_1^T measure. The above equation can be derived by using the relationship between the Brownian motions under \mathbb{R}_1^T and the Brownian motions under \mathbb{P} (as shown in Internet Appendix

Section I.D), as follows:

$$\begin{aligned} dW_{1s}^{\mathbb{R}^T} &= dW_{1s}^{\mathbb{P}} + [\mathbf{1}_{\{s \geq H\}}\gamma + \rho\sigma_r B_{\alpha_r}(T-s) - \mathbf{1}_{\{s < H\}}\rho\sigma_r B_{\alpha_r}(H-s)] ds, \\ dW_{2s}^{\mathbb{R}^T} &= dW_{2s}^{\mathbb{P}} + [\mathbf{1}_{\{s \geq H\}}\gamma_r + \sigma_r B_{\alpha_r}(T-s) - \mathbf{1}_{\{s < H\}}\sigma_r B_{\alpha_r}(H-s)] ds. \end{aligned}$$

Therefore, under the above specifications, the expected call option price $\mathbb{E}_t[C_H]$ is given by equation (40) with that $\mathbb{E}_t[P(H, T)]$ is derived in equation (IA4), and

$$\begin{aligned} \mathbb{E}_t[S_H] &= S_t \exp \left(\left(m_r + \gamma\sigma + \frac{\sigma_r^2}{2\alpha_r^2} + \frac{\rho\sigma\sigma_r}{\alpha_r} \right) (H-t) + (r_t - m_r) B_{\alpha_r}(H-t) \right. \\ &\quad \left. + \frac{\sigma_r^2}{2\alpha_r^2} [B_{2\alpha_r}(H-t) - 2B_{\alpha_r}(H-t)] - \frac{\rho\sigma\sigma_r}{\alpha_r} B_{\alpha_r}(H-t) \right), \end{aligned} \tag{IA11}$$

and

$$v_p = \sqrt{\left(\sigma^2 + \frac{\sigma_r^2}{\alpha_r^2} + \frac{2\rho\sigma\sigma_r}{\alpha_r} \right) (T-t) + \frac{\sigma_r^2}{\alpha_r^2} [B_{2\alpha_r}(T-t) - 2B_{\alpha_r}(T-t)] - \frac{2\rho\sigma\sigma_r}{\alpha_r} B_{\alpha_r}(T-t)}.$$

The assumption of a constant value for γ is not inconsistent with various multifactor asset pricing models, such as Merton's (1973a) ICAPM and Connor and Korajczyk's (1989) intertemporal APT. For example, under the Merton's (1973a) ICAPM framework, the market portfolio return and the projection of the asset return on the market portfolio return and the short rate process are given as follows:

$$\begin{aligned} \frac{dS_s^M}{S_s^M} &= (r_s + \gamma_M \sigma_M) ds + \sigma_M dW_{s,M}^{\mathbb{P}}, \\ \frac{dS_s}{S_s} &= \mathbb{E} \left[\frac{dS_s}{S_s} \right] + \beta_1 \left[\frac{dS_s^M}{S_s^M} - \mathbb{E} \left[\frac{dS_s^M}{S_s^M} \right] \right] + \beta_2 [dr_s - \mathbb{E}[dr_s]] + \sigma_e dW_{s,e}^{\mathbb{P}}, \end{aligned} \tag{IA12}$$

where γ_M is the Sharpe ratio or the market risk premium per unit of market return volatility, β_1 and β_2 are the systematic risk exposures of the asset with respect to mean-zero market return factor and mean-zero short rate change factor, $\sigma_e dW_{s,e}^{\mathbb{P}}$ is the idiosyncratic risk component of the asset return, and $W_{s,e}^{\mathbb{P}}$ is assumed to be independent of both $W_{s,M}^{\mathbb{P}}$ and $W_{2s}^{\mathbb{P}}$. The coefficients β_1 and β_2 are obtained as constants since all volatilities and correlations that these coefficients depend on are

assumed to be constants in equations (IA10) and (IA12). Under these assumptions, Merton's (1973a) ICAPM implies that the asset risk premium in equation (IA10) is:

$$\mu(s) - r_s = \gamma\sigma = \beta_1\gamma_M\sigma_M + \beta_2\gamma_r\sigma_r,$$

which implies that γ is a constant equal to:

$$\gamma = \beta_1\gamma_M\sigma_M/\sigma + \beta_2\gamma_r\sigma_r/\sigma.$$

This relation is also implied by the intertemporal APT of Connor and Korajczyk (1989) using the argument of diversification in large asset markets based upon the assumption of weakly correlated idiosyncratic terms, and the short rate process given by the Vasicek (1977) model.

VI. Expected Future Price of an Asset Exchange Option

While all options contracts written on tradable assets give the right to exchange one asset for another, the term “asset exchange options” is typically used for those option contracts whose strike price is a stochastic asset value instead of a constant. The asset S to be exchanged (as the strike price) is used as the numeraire for pricing these asset exchange options (e.g., see Margrabe (1978), Rady (1997), and examples in Vecer (2011)). We define the EMM corresponding to the numeraire S as the \mathbb{Q}^S measure, and the corresponding numeraire-specific EEM as the \mathbb{R}_1^S measure, given as a special case of the \mathbb{R}_1^* measure with the numeraire $G = S$. The definition of the EEM \mathbb{R}_1^S assumes that the asset S is any tradable asset other than the money market account and the T -maturity pure discount bond.

For simplicity of exposition, we use the Margrabe (1978) model to demonstrate the use of EEM \mathbb{R}_1^S for obtaining the expected future price of a call option that gives the option holder the right to exchange an asset $S = S_1$ for another asset S_2 . We extend the Margrabe (1978) model by allowing a stochastic short rate, and assume that the physical asset price processes S_1 and S_2 , and the physical short

rate process are given as follows:

$$\begin{aligned}\frac{dS_{1s}}{S_{1s}} &= (r_s + \gamma_1\sigma_1) ds + \sigma_1 dW_{1s}^{\mathbb{P}}, \\ \frac{dS_{2s}}{S_{2s}} &= (r_s + \gamma_2\sigma_2) ds + \sigma_2 dW_{2s}^{\mathbb{P}}, \\ dr_s &= \alpha_r (m_r - r_s) ds + \sigma_r dW_{rs}^{\mathbb{P}},\end{aligned}$$

where $W_1^{\mathbb{P}}$ and $W_2^{\mathbb{P}}$ are correlated Brownian motions with a correlation coefficient equal to ρ_{12} ; $W_r^{\mathbb{P}}$ is correlated with $W_1^{\mathbb{P}}$ and $W_2^{\mathbb{P}}$ with correlation coefficients equal to ρ_{1r} and ρ_{2r} , respectively; and γ_1 and γ_2 are defined as the risk premium per unit of volatility for S_1 and S_2 , respectively.

It is convenient to price the exchange option using equation (11), with $G = S_1$ as the numeraire asset, $\mathbb{R}_1^S = \mathbb{R}_1^S$ using this numeraire asset, $F_T = C_T = (S_{2T} - S_{1T})^+$ as the terminal payoff, and the numeraire values given as $G_T = S_{1T}$, and $G_H = S_{1H}$, as follows:

$$\begin{aligned}\mathbb{E}_t[C_H] &= \mathbb{E}_t^{\mathbb{P}} [S_{1H}] \mathbb{E}_t^{\mathbb{R}_1^S} \left[\frac{(S_{2T} - S_{1T})^+}{S_{1T}} \right] \\ &= \mathbb{E}_t^{\mathbb{P}} [S_{1H}] \mathbb{E}_t^{\mathbb{R}_1^S} [(V_T - 1)^+],\end{aligned}$$

where $V = S_2/S_1$ is the asset price ratio. Therefore, it suffices for us to obtain the process of V under the \mathbb{R}_1^S measure, which can be derived using Lemma 2 as follows:

$$\frac{dV_s}{V_s} = \mathbf{1}_{\{s < H\}} [\gamma_2\sigma_2 - \gamma_1\sigma_1 + (\rho_{2r}\sigma_2 - \rho_{1r}\sigma_1) \sigma_r B_{\alpha_r}(H - s)] ds + \sigma_2 dW_{2s}^{\mathbb{R}_1^S} - \sigma_1 dW_{1s}^{\mathbb{R}_1^S},$$

where $B_{\alpha_r}(\tau) = (1 - e^{-\alpha_r\tau})/\alpha_r$.

Note that since all volatilities, correlations, and drifts are either constant or deterministic in the above equation, V_T is lognormally distributed under the \mathbb{R}_1^S measure. Thus, using equation (14) and similar derivation logic of the expected future price of a call option under the Merton (1973b) model in Section III.A (given by equation (40)), the solution of the expected future price of the exchange

option is given as follows:

$$\mathbb{E}_t[C_H] = \mathbb{E}_t[S_{2H}]\mathcal{N}(\hat{d}_1) - \mathbb{E}_t[S_{1H}]\mathcal{N}(\hat{d}_2),$$

where

$$\hat{d}_1 = \frac{1}{v_p} \ln \frac{\mathbb{E}_t[S_{2H}]}{\mathbb{E}_t[S_{1H}]} + \frac{v_p}{2}, \quad \hat{d}_2 = \frac{1}{v_p} \ln \frac{\mathbb{E}_t[S_{2H}]}{\mathbb{E}_t[S_{1H}]} - \frac{v_p}{2},$$

with $v_p = \sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2)}(T - t)$, $\mathbb{E}_t[S_{jH}]$ is given similarly as in equation (IA11) with $(S_t, \gamma, \sigma, \rho)$ replaced with $(S_{jt}, \gamma_j, \sigma_j, \rho_{jr})$ for $j = 1, 2$.

It is important to note that while the valuation of an exchange option does not depend upon interest rates under the Margrabe (1978) model, the expected future price of the exchange option as given by the above equation *does depend upon the parameters of the interest rate process*. This dependence arises not from the hedge portfolio's integrated variance v_p^2 , but through the expected future prices $\mathbb{E}_t[S_{1H}]$ and $\mathbb{E}_t[S_{2H}]$.

VII. A Procedure to Extract the Expected FSPD

To obtain the expected FSPD from expected future prices of options using Theorem 3, we consider extending the methods used for obtaining the SPD (defined as the second derivative of the call price function with respect to the strike price in Breeden and Litzenberger (1978)) from current option prices. Both parametric estimation methods and non-parametric estimation methods can be used for extracting the SPD from option prices (Figlewski, 2018). The parametric methods include composite distributions based on the normal/lognormal (see, e.g., Jarrow and Rudd, 1982; Madan and Milne, 1994) and mixture models (see, e.g., Melick and Thomas, 1997; Soderlind and Svensson, 1997). The non-parametric estimation methods can be classified into three categories: maximum entropy (see, e.g., Stutzer, 1996; Buchen and Kelly, 1996), kernel (see, e.g., Ait-Sahalia and Lo, 1998; Pritsker, 1998), and curve-fitting (see, e.g., Jackwerth, 2000, 2004; Rosenberg and Engle, 2002; Figlewski, 2009) methods. We extend the fast and stable curve-fitting

method of Jackwerth (2004) that fits the implied volatilities using the Black and Scholes (1973) model, and then numerically approximates the SPD.⁴

We briefly recall the main steps for using the Jackwerth (2004) method. The first step inverts the option prices into Black-Scholes implied volatilities, $\bar{\sigma}(K/S_t)$, using a finite number of options with different strike prices. This allows greater stability since implied volatilities for different strike prices are much closer in size than the corresponding option prices. Let $\{K_1, K_2, \dots, K_N\}$ represent a set of strike prices ordered from the lowest to the highest. Using a “smoothness” parameter as in Jackwerth (2004), a smooth BS implied volatility (IV) curve $\hat{\sigma}(K/S_t)$ is fitted by optimization. In the next step, each optimized IV $\hat{\sigma}(K_i/S_t)$ is converted back into a call price using the Black-Scholes equation, that is, $C_i = C_{\text{BS}}(S_t, K_i, t, T, r; \hat{\sigma}(K_i/S_t))$, and the SPD (discounted \mathbb{Q}^T density) is estimated numerically as follows: For any $1 < i < N$,

$$\begin{aligned} f_t(S_T = K_i) &= P(t, T)p_t^{\mathbb{Q}^T}(S_T = K_i) \\ &= \frac{\partial^2 C_{\text{BS}}(S_t, K_i, t, T, r; \hat{\sigma}(K_i/S_t))}{\partial K_i^2} \\ &\approx \frac{C_{i+1} - 2C_i + C_{i-1}}{(\Delta K)^2}. \end{aligned}$$

We extend the fast and stable curve-fitting method of Jackwerth (2004) for obtaining the expected FSPD from the expected future option prices by not only fitting implied volatilities using the current call price formula of the Black and Scholes (1973) model, but also the *implied drifts* using the expected future call price formula of the Black and Scholes (1973) model given in equation (C1), and then numerically approximate the expected FSPD using Theorem 3 (equation (43)). This extension is outlined in the following steps. The expected returns of options are first estimated using historical data, and then expected option prices can be obtained for a finite number of strike prices given by the set $\{K_1, K_2, \dots, K_N\}$. As in Jackwerth (2004), the current option prices are inverted into BS IV $\bar{\sigma}(K_i/S_t)$ and using a “smoothness” parameter, a smooth BS IV curve $\hat{\sigma}(K/S_t)$ is fitted by

⁴For an extensive review of the curve-fitting methods, see Jackwerth (2004) and Figlewski (2018).

optimization. Then, using fixed optimized IV $\hat{\sigma}(K/S_t)$, we invert the expected future option price into the BS implied drift $\bar{\mu}(K/S_t)$ using the Black-Scholes expected future call price in equation (C1). Next, following the optimization method of Jackwerth (2004), a smooth ID (implied drift) curve $\hat{\mu}(K/S_t)$ is fitted by solving a similar optimization problem that balances smoothness against the fit of the IDs. For each strike price, the optimized IV $\hat{\sigma}(K_i/S_t)$ and ID $\hat{\mu}(K_i/S_t)$, are converted back into the expected call price using the Black-Scholes expected future price in equation (C1), that is, $EC_i = \hat{C}_{\text{BS}}(S_t, K_i, t, T, H, r; \hat{\sigma}(K_i/S_t), \hat{\mu}(K_i/S_t))$. Finally, the expected FSPD is numerically approximated using Theorem 3 (equation (43)), as follows: For any $1 < i < N$,

$$\begin{aligned} g(H, S_T = K_i) &= \mathbb{E}_t^{\mathbb{P}} [P(H, T)] p_t^{\mathbb{R}^T}(S_T = K_i) \\ &= \frac{\partial^2 \hat{C}_{\text{BS}}(S_t, K_i, t, T, H, r; \hat{\sigma}(K_i/S_t), \hat{\mu}(K_i/S_t))}{\partial K_i^2} \\ &\approx \frac{EC_{i+1} - 2EC_i + EC_{i-1}}{(\Delta K)^2}. \end{aligned}$$

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